A linear logic modulo

Olivier Hermant

PPS, Université Denis Diderot, 2 place Jussieu, 75251 Paris Cedex 05, France
hermant@pps.jussieu.fr
http://www.pps.jussieu.fr/˜hermant

Abstract. We describe the rules of linear logic modulo and we prove its soundness/completeness wrt phase semantics. Then we prove cut elimination for some conditions on rewrite rules, some of which are new (positivity/negativity). At last, we give hints of proofs nets modulo.

1 Introduction

This could be useful for *dynamic resources*, that are generated along the proof, after an instanciation for example.

2 The syntax

2.1 Language

The terms are constructed with a denumerable set of variables $\mathcal{V}$ and an untyped signature Σ containing function symbols together with an arity. Well former terms are inductively defined: a variable is a well formed term and an applied function symbols to *arity* well formed terms is well formed. One could in the usual way extend this definition to a typed signature.

On the same way, we form propositions with a signature Σ' containing n-ary predicate symbols, among which $\mathbf{1}/0, \mathbf{0}/0, \top/0, \bot/0$, the Linear Logic connectors: $\&/2$, $⅋/2, \otimes/2, \oplus/2, \multimap/2$, $!/1, ?/1$ and the quantifiers $\forall, \exists$ of rank $\langle \mathcal{V}, prop, prop\rangle$. A well-formed formula is defined inductively: it is either an applied predicate symbol $P(t_1, ..., t_n)$ or the negation of it $P(t_1, ..., t_n)^\perp$, a combination of well-formed formulas with the help of the binary connectors or a quantification over a variable on a well-formed formula. We allow negation only on predicates, and we work under de Morgan's laws and classical equivalences:

This will allow us to consider sequents with only right-hand side members: left-hand side becomes negated when passing to the right hand side, as usual.

At last, we use the standard notions for free, fresh and bound variables, atomic formula and sequent, which is a pair of multisets of formulas (order does not matter).

2.2 Rewrite rules

As usual in deduction modulo, we use rewrite rules on terms and on atomic formulas:

$$
\begin{array}{cc}
\top^\perp = \mathbf{0} & \mathbf{0}^\perp = \top \\
\bot^\perp = \mathbf{1} & \mathbf{1}^\perp = \bot \\
\multicolumn{2}{c}{(A^\perp)^\perp = A} \\
\multicolumn{2}{c}{A \multimap B = A^\perp \parr B} \\
(\forall x A)^\perp = \exists x A^\perp & (\exists x A)^\perp = \forall x A^\perp \\
(A \otimes B)^\perp = A^\perp \parr B^\perp & (A \parr B)^\perp = A^\perp \otimes B^\perp \\
(A \oplus B)^\perp = A^\perp \,\&\, B^\perp & (A \,\&\, B)^\perp = A^\perp \oplus B^\perp \\
(!A)^\perp = ?(A^\perp) & (?A)^\perp = !(A^\perp)
\end{array}
$$

Fig. 1. connector equivalences in classical linear logic

Definition 1 (rewrite rules). *a rewrite rule $l \to r$ is a pair of terms or formulas such that:*

- *the variables if r are included in those of l.*
- *if l is a formula, it is atomic.*

A formula P rewrites to a formula Q (notation $P \to Q$) with if there exists a rewrite rule $l \to r$, and a substitution σ such that σl appears at an occurence o in P and $Q = P[\sigma r]_o$.

$\to^*$ is the reflexive-transitive closure of rrule and $\equiv$ is the reflexive-symmetric-transitive closure of $\to$. $P\!\downarrow$ represents the normal form of P for $\mathcal{R}$, if it exists.

In the rest of this paper, one consider some set $\mathcal{R}$ of rewrite rule.

2.3 Deduction rules

In order to integrate rewriting into linear logic, one modify the rules as follow:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta} \multimap\text{-r}$$

becomes

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash C, \Delta} \multimap\text{-r iff } C \equiv A \multimap B$$

The intended meaning is that C should "simplify" in $A \multimap B$ (so, from a computer point of view, one must compute the value of C). Notice also that rewriting is not trivial in the following sense: after a $\forall$-g or a $\exists$-r rule, the formula produced can be reducible for $\to$ even if the previous formula is not.

The rules of linear logic we are going to enhance with rewriting are the classical linear logic rules, that can be found, for instance in [8, 4]. The rules of linear logic modulo are summed up in figure 2 for the one-sided sequent calculus. As usual, we can restrict the axiom rule (init) to be atomic, i.e. $\overline{\vdash A^\perp, A}$ is a proof if and only if A is atomic. This is not a restriction, since any proof using init with non atomic formulas can easily be transformed to satisfy this constraint.

Identities

$$\frac{}{\vdash A^{\perp}, A}\ \text{init} \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}, A \equiv B^{\perp}$$

Multiplicatives

$$\frac{}{\vdash A}\ \mathbf{1}\text{-r}, A \equiv \mathbf{1} \qquad \frac{\vdash \Delta}{\vdash A, \Delta}\ \bot\text{-r}, A \equiv \bot$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash C, \Gamma, \Delta}\ \otimes\text{-r}, C \equiv A \otimes B \qquad \frac{\vdash A, B, \Delta}{\vdash C, \Delta}\ ⅋\text{-r}, C \equiv A ⅋ B$$

Additives

no $\mathbf{0}$-r $\qquad \frac{}{\vdash \top, \Gamma}\ \top\text{-r}, A \equiv \top$

$$\frac{\vdash A, \Delta \qquad \vdash B, \Delta}{\vdash C, \Delta}\ \&\text{-r}, C \equiv A \& B \qquad \frac{\vdash A, \Delta}{\vdash C, \Delta}\ \oplus\text{-r1}, C \equiv A \oplus B \qquad \frac{\vdash B, \Delta}{\vdash C, \Delta}\ \oplus\text{-r2}, C \equiv A \oplus B$$

Quantifiers

$$\frac{\vdash A, \Delta}{\vdash C, \Delta}\ \forall\text{-r}, C \equiv \forall x A,\ x \text{ fresh} \qquad \frac{\vdash (t/x)A, \Delta}{\vdash C, \Delta}\ \exists\text{-r}, C \equiv \exists x A,\ t \text{ any term}$$

Exponentials

$$\frac{\vdash A, B, \Delta}{\vdash C, \Delta}\ \text{contraction}, A \equiv B \equiv C \equiv ?D \qquad \frac{\vdash \Delta, A}{\vdash \Delta, B}\ \text{dereliction}, B \equiv ?A$$

$$\frac{\vdash \Delta}{\vdash \Delta, B}\ \text{weakening}, B \equiv ?A \qquad \frac{\vdash \Delta, A}{\vdash \Delta, B}\ \text{promotion}, B \equiv !A \text{ and } \Delta \equiv ?\Gamma$$

Fig. 2. Inference rules of Linear Logic modulo

3 The semantics: phase spaces

We consider first-order phases spaces, as for instance defined in [8], to which we add the condition that it should be a model of the rewrite rules. In the follwing definition, we make the following abuse of notation: linear connector and constant symbols are used also for phase space operations and sets. This abuse is obviously driven by the soundness theorem we are going to prove in next section.

Definition 2 (Classical phase space ([8])). *Let $(M, .)$ be a commutative monoid, 1 be its unit and $\perp$ be a fixed subset of M.*

For any $\alpha \subseteq M$ we define the set $\alpha^\perp$ as $\{a \mid \alpha.a \subseteq \perp\}$. A fact is a set $\alpha \subseteq M$ such that $\alpha = \alpha^{\perp\perp}$.

Let $J \subseteq M$ be a submonoid of M such that $a \in J$ implies $\{a\}^{\perp\perp} \subseteq \{a.a\}^{\perp\perp}$. We let, for any α, β facts of M:

- $\top = M$
- $\mathbf{1} = \perp^\perp = \{b \mid \forall a \in \perp, a.b \in \perp\} = \{1\}^{\perp\perp}$
- $\mathbf{0} = \top^\perp = \{a \mid M.a \subseteq \perp\}$
- $\alpha \mathbin{\&} \beta = \alpha \cap \beta$
- $\alpha \oplus \beta = (\alpha \cup \beta)^{\perp\perp}$
- $\alpha \otimes \beta = (\alpha.\beta)^{\perp\perp}$
- $\alpha \parr \beta = (\alpha^\perp.\beta^\perp)^\perp$
- $I = \mathbf{1} \cap J$
- $!\alpha = (I \cap \alpha)^{\perp\perp}$
- $?\alpha = (!\alpha^\perp)^\perp = (I \cap \alpha^\perp)^\perp$

We note $D_M \subseteq \mathfrak{P}(M)$ the set of facts of M. A triple $\langle D, I, \perp\rangle$ is called a (classical) phase space if $\perp \in D \subseteq D_M$ and D is closed under all the operators above. If moreover D is closed by arbitratry intersections, the phase space is said to be complete.

Notice that all the operations defined in Def. 2transform facts into facts, as shown in the following lemma. As an exercise, one can prove the following properties:

Lemma 1. *For any $\alpha, \beta \subseteq M$:*

- *$\alpha \subseteq \alpha^{\perp\perp}$,*
- *$(\alpha^\perp)^{\perp\perp} \subseteq \alpha^\perp$ ($\alpha^\perp$ is a fact),*
- *$(\alpha^{\perp\perp})^{\perp\perp} \subseteq \alpha^{\perp\perp}$,*
- *$\alpha \subseteq \beta$ implies $\beta^\perp \subseteq \alpha^\perp$.*
- *$\alpha \subseteq \beta$ implies $\alpha^{\perp\perp} \subseteq \beta^{\perp\perp}$,*
- *$\alpha^{\perp\perp}.\beta^{\perp\perp} \subseteq \alpha.\beta^{\perp\perp}$,*
- *$\alpha^{\perp\perp}$ is the smallest fact including α,*
- *$(\alpha \cup \beta)^{\perp\perp} = (\alpha^\perp \cap \beta^\perp)^\perp$,*
- *$\alpha \cap \beta$ is a fact if α, β are facts.*

Definition 3 (phase structure). *Let $\mathcal{L}$ be language with function and predicate symbols. A phase structure is a tuple $\langle S, D, I, \perp, \hat{}\,\rangle$ where $\langle D, I, \perp\rangle$ is a phase space, S a base set (the domain) and $\hat{}$ is an assignment assigning to each predicate symbol P of arity n a function $\hat{P} : S^n \to D$ and to each function symbol f of arity n a function $\hat{f} : S^n \to S$. A phase structure is complete when the undelying phase space is complete.*

A phase structure gives the interpretation of every basic objects. It gives rise to a unique model by a straightforward extension to comound objects:

Definition 4 (Phase models). *Let $\langle D, I, \perp\rangle$ be a complete phase structure, let S be a set and ϕ be an assignment associating to every variable an element of the same type in the domain. We define the interpretation of terms t^*_ϕ and formulas A^*_ϕ by induction over their structures:*

- $x^*_\phi = \phi(x)$
- $(f(t_1, ..., t_n))^*_\phi = \hat{f}((t_1)^*_\phi, ..., (t_n)^*_\phi)$

and for formulas:

- $(P(t_1, ..., t_n))^*_\phi = \hat{P}((t_1)^*_\phi, ..., (t_n)^*_\phi)$
- $(A^\perp)^*_\phi = (A^*_\phi)^\perp$
- $(A \otimes B)^*_\phi = A^*_\phi \otimes B^*_\phi$
- $(A \parr B)^*_\phi = A^*_\phi \parr B^*_\phi$
- $(A \,\&\, B)^*_\phi = A^*_\phi \,\&\, B^*_\phi$
- $(A \oplus B)^*_\phi = A^*_\phi \oplus B^*_\phi$
- $(!A)^*_\phi = !A^*_\phi$
- $(?A)^*_\phi = ?A^*_\phi$
- $(\forall x A)^*_\phi = \bigcap\{A^*_{\phi+(d/x)} \mid d \in S\}$
- $(\exists x A)^*_\phi = (\forall x A^\perp)^{*\perp}_\phi = \left(\bigcup\{A^*_{\phi+(d/x)} \mid d \in S\}\right)^{\perp\perp}$

A tuple $\langle S, D, I, \perp, \phi\rangle$ is called a (first order) phase model. A formula is said valid *in this model if and only if* $1 \in A^*_\phi$

A phase model is a model of the rewrite system $\mathcal{R}$ if for any formulas such that $A \equiv B$, $A^*_\phi = B^*_\phi$ holds.

Notice that only the assignment of bound variables can change. We will forget the underscript ϕ when it is not ambiguous.

4 Soundness

The confluence property is there only to ensure the following proposition:

Proposition 1. *Two non atomic equivalent formulas modulo $\mathcal{R}$ have the same main connector.*

Proof. See for instance [6].

With respect to these settings, proving the soundness theorem is mere routine:

Theorem 1 (Soundness). *Let $A_1, ..., A_n$ be formulas, let $\mathcal{R}$ be a confluent rewrite system. If a sequent $\vdash A_1, ..., A_n$ is derivable in Linear Logic modulo then $1 \in (A_1)^* \parr ... \parr (A_n)^*$ for any phase model of $\mathcal{R}$.*

Proof. By induction on the structure of the derivation, considering every inference rule.

5 Completeness and cut elimination

At this point, we should try to prove the completeness theorem:

Theorem 2 (Completeness). *Let $\mathcal{R}$ be a rewrite system. If a formula is valid in any phase model, then it is derivable in Linear Logic modulo $\mathcal{R}$.*

Instead of proving this (mainly: contructing some variant of a Lindenbaum's algebra) we will follow another way, trying to prove *strong* completeness, following the method of [8]. The benefit will be immediate: a cut elimination theorem for linear logic modulo. But here we will need some conditions on the rewrite system $\mathcal{R}$, since it is well-know ([3, 5]) that in deduction modulo some rewrite systems, the cut rule is not eliminable (nor normalizable). So, under some proviso on the rewrite system $\mathcal{R}$ considered, we will prove the:

Theorem 3 (Strong completeness theorem). *Let $\mathcal{R}$ a rewrite system, let A be a formula. If $1 \in A^*$ in any phase model, then $\vdash A$ is provable.*

For this purpose, we construct a canonical phase *space* based on contexts and *cut free* provability. Let a signature Σ, we let M be the set of all contexts, i.e. the set of finite sets of formulas, where multiple occurence of a formula of the type $?A$ is counted only once, so that $\{?A, ?A\} = \{?A\}$. The . operation is concatenation and its unit $1 = \emptyset$. J is $?M$ that is the set of contexts of the form $\{?A_1, ..., ?A_n\}$.

Definition 5 (Outer value ([8])). *Let A be a formula. We define its outer value $[\![A]\!] = \{\Delta \mid\ \vdash^* A, \Delta\}$, i.e. the set of contexts proving A without cut.*

Remark 1. Obviously, if $A \equiv B$, $[\![A]\!] = [\![B]\!]$.

We now define $\perp$ as $[\![\perp]\!] = \{\Delta \mid\ \vdash \Delta$ is provable $\}$, and prove that outer values are facts:

Lemma 2. *(i)* $\{A\}^\perp = [\![A]\!]$
(ii) $A \in [\![A]\!]^\perp$.
(iii) $[\![A]\!]^{\perp\perp} = [\![A]\!]$ *($[\![A]\!]$ is a fact).*

Proof. *(i)* The definition of $\{A\}^\perp$ is $\{\Gamma \mid\ \vdash \Gamma, \Delta$ for any $\Delta \in \{A\}\}$. But $\{A\}$ contains only A.
(ii) $\{A\} \subseteq \{A\}^{\perp\perp} = [\![A]\!]^\perp$ by Lem. 1 and *(i)*.
(iii) from Lem. 1, $\{A\}^\perp$ is a fact. We conclude by *(i)*.

We let D be the set of all facts. It is closed by arbitrary intersections, as proved by Girard [4]:

$$\bigcap_{i \in I} \alpha_i = \bigcap_{i \in I} {\alpha_i}^{\perp\perp} = \left(\bigcup_{i \in I} {\alpha_i}^{\perp}\right)^{\perp}$$

D is closed by intersection and the other operators, hence $\langle D, I, \perp \rangle$ is a phase space.

Note that we only defined a phase space, not a phase model, and we have a good reason for that. A degree of freedom is left: how to interpret function and proposition symbols ? (And what domain can we choose ?) This is all the trick of deduction modulo. We will answer this question in different ways, depending on the rewrite system we consider.

5.1 An order condition

Besides the confluence of the rewrite systemt $\mathcal{R}$, we suppose, as in [7, 6] that we have a well-founded order $\succ$ such that:

- if B is a proper subformula of A then $A \succ B$,
- if $A \rightarrow^* B$ then $A \succ B$.

We define the phase model by induction over $\succ$:

Definition 6. *We let the set S be the set of all ground terms of the language. Let ϕ be an assignment in S for variables. For any function symbol we let:*

$$\hat{f} : (t_1, ..., t_n) \in S^n \mapsto f(t_1, ..., t_n)$$

And we define by mutual induction ϕ on predicates and the interpretation of formulas $A \mapsto A^*_\phi$:

$$\hat{P} : (t_1, ..., t_n) \in S^n \mapsto \begin{cases} [\![P(t_1, ..., t_n)]\!] \text{ if } P(t_1, ..., t_n) \text{ is in normal form} \\ (P(t_1, ..., t_n)\downarrow)^*_\phi = (P(t_1, ..., t_n)\downarrow)^* \text{ otherwise} \end{cases}$$

The interpretation of formulas is defined as in Def. 4.

Remark 2. Thanks to $\succ$, Def. 6 is well founded. Moreover, we really define a model interpretation, giving explicit interpretation only for predicates. In the second part of the definition of $\phi(P)$ notice also that since $P(t_1, ..., t_n)\downarrow$ is a ground formula, we do not need ϕ in $(P(t_1, ..., t_n)\downarrow)^*_\phi$. At last, normal forms exist since $\succ$ is well-founded.

Things work nice also because of the following lemmas.

Lemma 3. *Let t be a term. Defining σ as the subsitution associating $\phi(x)$ to x, in the phase model of Def. 6 $t^*_\phi = \sigma t$.*

Proof. The first statement is proved by a straightforward induction on the term structure. The second statement is a direct consequence of the first.

We check that this model is a model of $\mathcal{R}$. this should be rather intuitive, from Def. 6.

Lemma 4. *For any formula A, any assignment ϕ, $A^*_\phi = ((\sigma A)\downarrow)^*_\phi$, where σ is the substitution associated to ϕ.*

Proof. By induction over the w.f.o. $\succ$. Let see some key cases:

- $\sigma A = \sigma P(t_1, ..., t_n)$ is a normal atom for $\mathcal{R}$. Then we have, from Lem. 3 and Def. 6:

$$A^*_\phi = \hat{P}((t_1)^*_\phi, ..., (t_n)^*_\phi) = \hat{P}(\sigma t_1, ..., \sigma t_n) = [\![\sigma A]\!] = (\sigma A)^*$$

- $\sigma A = \sigma P(t_1, ..., t_n)$ is an atom that is not normal for $\mathcal{R}$. Then, using Def. 6 and Lem. 3 we can derive the following equalities. Each of them is trivial, but needs to be carefully checked:

$$(A)^*_\phi = \hat{P}((t_1)^*_\phi, ..., (t_n)^*_\phi) = \hat{P}(\sigma t_1, ..., \sigma t_n) = (P(\sigma t_1, ..., \sigma t_n)\downarrow)^* = ((\sigma A)\downarrow)^*$$

– σA is $\sigma\forall xB = \forall x(\sigma B)$, supposing without loss of generality that $\sigma(x) = x$. Then we have the following equalities:

$$A^*_\phi = (\forall xB)^*_\phi = \bigcap_{t\in S}(B)^*_{\phi+(t/x)} = \bigcap_{t\in S}((\sigma' B)\downarrow)^*$$

last equality by induction hypothesis, since $\forall x\sigma B \succ \sigma' B$ with $\sigma' = \sigma + (t/x)$ for any $t \in S$. One can continue:

$$\bigcap_{t\in S}((\sigma' B)\downarrow)^* = \bigcap_{t\in S}(((t/x)((\sigma B)\downarrow))\downarrow)^* = \bigcap_{t\in S}((t/x)((\sigma B)\downarrow)))^*$$

since σ is a ground substitution. The last equality is obtained by induction hypothesis, since $\forall x(\sigma B) \succ (t/x)(\sigma B) \succ (t/x)((\sigma B)\downarrow)$. Hence, we get:

$$(\forall xB)^*_\phi = \bigcap_{t\in S}((t/x)\,((\sigma B)\downarrow)))^* = (\forall x(\sigma B)\downarrow)^*_\phi = ((\sigma\forall xB)\downarrow)^*$$

that is what we wanted to show.
– the existential case follows exactly the same pattern. The connector and constant cases are much more simpler, since rewriting interacts only with instantiation.

As a consequence we get the corollary:

Lemma 5. *The model of Def. 6 is a model of $\mathcal{R}$.*

Proof. Let $A \equiv B$ two propositions. By Lem. 4, for any assignment ϕ we have:

$$A^*_\phi = ((\sigma A)\downarrow)^*_\phi = ((\sigma B)\downarrow)^*_\phi = B^*_\phi$$

since the normal form of A and B exists (the order $\succ$ is well-founded) and the rewrite system is confluent.

Then we can prove the main lemma:

Lemma 6 (Main lemma). *Let A be a formula, ϕ an assignment and σ the substitution associated to it. Then:*

$$\sigma A^\perp \in A^*_\phi \subseteq [\![\sigma A]\!]$$

Proof. By induction on the w.f.o. $\succ$, making cases on σA. The normal atomic case is an equality indeed. If we encounter a non normal atomic formula, we normalize it and this makes the order $\succ$ decrease. The connector cases is exactly the same induction as [8]. TODO HERE !!! DETAILS !!!

5.2 A positivity condition

We suppose that the rewrite system is *positive*:

Definition 7 (Positivity condition). *A rewrite system $\mathcal{R}$ is said to be positive if there exists a partition of the predicate symbols into positive and negative, such that for any propositional rewrite rule*

$$P(t_1, ..., t_n) \to A \in \mathcal{R}$$

– *if P is* positive*, so is A.*
– *if P is* negative*, so is A.*

Where positivity and negativity is defined as follows:

Definition 8 (Positivity of occurences). *Given a partition of the predicate symbols into* positives *and* negatives *one, one say that an occurence of an atomic formula P is said to be* positive *(resp.* negative*) in a formula A if:*

– *A is P and P is positive (resp. negative).*
– *$A = B^{\perp}$ and the occurence of P is negative (resp. positive) in B.*
– *$A = B \otimes C$ and the occurence of P is positive (resp. negative) in B or in C.*
– *and so on for the other connectors.*

A formula A is said positive *(resp.* negative*) if atoms occur only positively (resp. negatively) in A.*

Of course, on could invert negatives and positives. The point is to find a partition and we will consider such a partition in the remaining of this paper. We hence consider in this section a rewrite system $\mathcal{R}$ that is confluent and positive. Then we define the following model:

Definition 9. *We let the set S be the set of all ground terms of the language. We set:*

$$\hat{f} : (t_1, ..., t_n) \in S^n \mapsto f(t_1, ..., t_n)$$

And we define $\phi(P)$ for a predicate symbol P:

$$\hat{P} : (t_1, ..., t_n) \mapsto \begin{cases} [\![P(t_1, ..., t_n)]\!] & \text{if } P \text{ is positive} \\ \{P(t_1, ..., t_n)^{\perp}\}^{\perp\perp} & \text{if } P \text{ is negative} \end{cases}$$

At last we define the interpretation function as in Def. 4.

Notice. Notice that $[\![P]\!] = \{P\}^{\perp} \supseteq \{P^{\perp}\}^{\perp\perp}$ is provable, but the converse is not (unless we allow the use of the cut rule).

Notice that we defined the interpretation of atomic predicates without the help of the rewrite system.

Def. 9 surely defines a phase model, but is this a phase model of the rewrite system $\mathcal{R}$? The answer is yes, due to the positivity condition. What we are going to prove is that every rewrite rule is valid in the sense that if $P \to A$ is a rewrite rule, then $A^*_{\phi} = P^*_{\phi}$. This is a consequence of the following lemma:

Lemma 7. *Let A be a formula with free variable, and ϕ an assignment of ground terms to variables (also considered as a ground substitution, by abuse of language).*

If A is positive (resp. negative), then $A^*_{\phi} = [\![\phi A]\!]$ (resp. $A^*_{\phi} = \{(\phi A)^{\perp}\}^{\perp\perp}$).

Proof. By induction over the proposition structure. We forget the subscript ϕ when it is not important (ie everywhere but in the quantifier cases).

Let's note P^+ a positive predicate (or formula) and P^- a negative predicate (or formula). Moreover, $\mathbf{t}$ denotes a vector of terms, and $P(\mathbf{t})$ should be understood as $P(t_1, ..., t_n)$.

- if A is an atomic predicate $P(\mathbf{t})$, then it is by definition.
- if $A = B^{\perp}$ then the result follows from *(i)* of Lem. 2. If A is positive, then by induction hypothesis, we have $(B^{-})^{*} = \{B^{\perp}\}^{\perp\perp}$. Hence $(A^{+})^{*} = \{B^{\perp}\}^{\perp} = A^{\perp} = [\![A]\!]$. Conversely, if A is negative then by induction hypothesis we have: $(B^{-})^{*} = [\![B]\!] = \{B\}^{\perp}$. Hence $A^{*} = \{B\}^{\perp\perp} = \{A^{\perp}\}^{\perp\perp}$.
- if $A = B \mathbin{\&} C$. Consider the positive case. B and C are positive. Hence $B^{*} = [\![B]\!]$ and $C^{*} = [\![C]\!]$, and $A^{*} = [\![B]\!] \cap [\![C]\!]$. Let $\Gamma \in A^{*}$, then the derivation:

$$\frac{\vdash \Gamma, B \qquad \vdash \Gamma, C}{\vdash \Gamma, B \mathbin{\&} C}$$

justifies the fact that $\Gamma \in [\![A]\!]$. Conversely, let $\Gamma \in [\![A]\!]$. Then we have a cut free proof of π of $\vdash \Gamma, A$. Then, we replace in π this particular occurence of A by B, and this gives us almost a proof of $\vdash \Gamma, B$, since the init rule is restricted to hold between axioms: we just need to erase all the occurence in π of the $\&$-r rule on A, keeping only the proof of the left premise, which is a proof of $\vdash \Gamma, B$. In the same way, we have also a cut free proof of $\vdash \Gamma, C$. Hence $\Gamma \in [\![B]\!] \cap [\![C]\!]$. To sum up, $A^{*}[\![B]\!] \cap [\![C]\!] = [\![A]\!]$.

If A is negative, by induction hypothesis we have $\{B\}^{\perp\perp} \cap \{C\}^{\perp\perp} = B^{*} \cap C^{*} = A^{*}$ and we mut show that this is equal to $\{A\}^{\perp\perp}$. Let us first show that $\{A\}^{\perp\perp} \subseteq \{B\}^{\perp\perp} \cap \{C\}^{\perp\perp}$. For this, we first show $\{A\}^{\perp\perp} \subseteq \{B\}^{\perp\perp}$ or equivalently: $\{B^{\perp}\}^{\perp} \subseteq \{A^{\perp}\}^{\perp}$. So, let $\Gamma \in \{B^{\perp}\}^{\perp}$. We have a cut free proof of $\vdash \Gamma, B^{\perp}$. And the inference rule:

$$\frac{\vdash \Gamma, B^{\perp}}{\vdash \Gamma, B^{\perp} \oplus C^{\perp}}$$

Shows that we also have a cut free proof of $\vdash \Gamma, A^{\perp}$. Hence $\Gamma \in \{A^{\perp}\}^{\perp}$, and therefore $\{B^{\perp}\}^{\perp} \subseteq \{A^{\perp}\}^{\perp}$. We prove exactly in the same way that $\{C^{\perp}\}^{\perp} \subseteq \{A^{\perp}\}^{\perp}$. Thus the first inclusion is proved.

Conversely, we show $\{B\}^{\perp\perp} \cap \{C\}^{\perp\perp} \subseteq \{A\}^{\perp\perp}$. For this, let $\Gamma \in \{B\}^{\perp\perp} \cap \{A\}^{\perp\perp}$. This means that for any Δ' such that $\vdash \Delta', B$ or $\vdash \Delta', C$ have a cut free proof, we have a cut free proof of $\vdash \Gamma, \Delta'$. We have to show that $\Gamma \in \{A\}^{\perp\perp}$, i.e. for any $\Delta \in \{A^{\perp}\}^{\perp}$ (Δ is such that $\vdash \Delta, A^{\perp}$ has a cut free proof) $\vdash \Gamma, \Delta$ has a ccut free proof.
So, let Δ such that $\vdash \Delta, A^{\perp}$ has a cut free proof. We prove that $\vdash \Delta, B^{\perp}$ and $\vdash \Delta, C^{\perp}$ both have a cut free proof, and that allows us to conclude since $\Gamma \in \{B\}^{\perp\perp} \cap \{C\}^{\perp\perp}$. Let π be thee proof of $\vdash \Delta, A^{\perp}$ which is also a proof of $\vdash \Delta, B^{\perp} \oplus C^{\perp}$. We construct a proof π' of $\vdash \Gamma, \Delta$ by replacing every occurence of $B^{\perp} \oplus C^{\perp}$ by Γ and, when a $\oplus$-r1 rule (resp. a $\oplus$-r2 rule) on $B^{\perp} \oplus C^{\perp}$ is met – which must be the case since the init rule is on atomic predicates only – we use the fact that $\Gamma \in \{B\}^{\perp\perp}$ (resp. $\Gamma \in \{C\}^{\perp\perp}$), ie since we have a cut free proof of some $\vdash \Delta', B$, we have a direct proof of $\vdash \Gamma, \Delta'$. Hence $\Gamma \in \{A\}^{\perp\perp}$, and we can conclude that $\{B\}^{\perp\perp} \cap \{C\}^{\perp\perp} \subseteq \{A\}^{\perp\perp}$.
- if $A = B \otimes C$. Consider the positive case. Unsing induction hypothesis on B^{*} and C^{*} one can restrict the equality to prove to $([\![B]\!].[\![C]\!])^{\perp\perp} = [\![A]\!]$.
For the direct inclusion, we show $[\![B]\!].[\![C]\!] \subseteq [\![A]\!]$ and conclude by applying biorthogonal, with the help of Lem. 1 and 2. Let Γ and Δ such that we have cut free proofs of $\vdash \Gamma, B$ and $\vdash \Delta, C$. the following inference:

$$\frac{\vdash \Gamma, B \qquad \vdash \Delta, C}{\Gamma, \Delta, B \otimes C}$$

shows that $\Gamma, \Delta \in [\![A]\!]$.
For the reverse inclusion, let $\Gamma \in [\![A]\!]$. Then we have a cut free proof of $\vdash \Gamma, B \otimes C$. We want to show $\Gamma \in ([\![B]\!].[\![C]\!])^{\perp\perp}$. Let $\Delta \in ([\![B]\!].[\![C]\!])^{\perp}$, ie such that $\vdash \Delta, \Gamma_B, \Gamma_C$ has a cut free proof for any $\Gamma_B \in [\![B]\!]$ and $\Gamma_C \in [\![C]\!]$. One must show that $\vdash \Gamma, \Delta$ has a cut free proof. We construct this proof by induction on the proof of $\vdash \Gamma, B \otimes C$. We as usual replace $B \otimes C$ by Δ and applying (inductively) exactly the same rules, until we meet (and we must meet by the atomicity of init) the $\otimes$-r rule on $B \otimes C$. We have a cut free proof of the two premises $\vdash \Gamma_B, B$ and $\vdash \Gamma_C, C$, with $\Gamma = \Gamma_B, \Gamma_C$, then we know by hypothesis on Δ that we have a cut free proof of $\vdash \Gamma, \Delta$.

Consider now the negative case. We must show: $\{(B \otimes C)\}^{\perp\perp} = \left(\{B\}^{\perp\perp}.\{C\}^{\perp\perp}\right)^{\perp\perp}$ or, equivalently:

$$\left(\{B\}^{\perp\perp}.\{C\}^{\perp\perp}\right)^{\perp} = \{B^{\perp} \parr C^{\perp}\}^{\perp}$$

For the direct inclusion, let $\Gamma \in \left(\{B\}^{\perp\perp}.\{C\}^{\perp\perp}\right)^{\perp}$. Since $B^{\perp} \in \{B\}^{\perp\perp}$ and $C^{\perp} \in \{C\}^{\perp\perp}$, we have a cut free proof of $\vdash \Gamma, B^{\perp}, C^{\perp}$. Hence, the following inference:

$$\frac{\vdash \Gamma, B^{\perp}, C^{\perp}}{\vdash \Gamma, B^{\perp} \parr C^{\perp}}$$

shows that $\Gamma \in \{B^{\perp} \parr C^{\perp}\}^{\perp}$.
Conversely, let $\Gamma \in \{B^{\perp} \parr C^{\perp}\}^{\perp}$. Let also $\Delta_B \in \{B\}^{\perp\perp}$ and $\Delta_C \in \{C\}^{\perp\perp}$. We have to show that we have a cut free proof of $\vdash \Gamma, \Delta_B, \Delta_C$. We construct this proof by induction on the proof π of $\vdash \Gamma, B^{\perp} \parr C^{\perp}$, applying inductively the same rules, until we meet the $\parr$-r rule. Then we have a proof of the premise $\vdash \Gamma', B^{\perp}, C^{\perp}$. This means that $\Gamma' \cup \{B^{\perp}\} \in \{C^{\perp}\}^{\perp}$. Therefore, we have by hypothesis on Δ_C a cut free proof of $\vdash \Gamma', B^{\perp}, \Delta_C$. This shows that $\Gamma' \cup \Delta_C \in \{B^{\perp}\}^{\perp}$. Hence, by hypothesis again, we have a proof of the sequent $\vdash \Gamma', \Delta_B, \Delta_C$.

- if $A = \forall x B(x)$. Suppose A is positive. Using induction hypothesis, it is sufficient to show: $\bigcap\{[\![(\phi + (t/x))B(x)]\!], t \text{ ground }\} = [\![\phi \forall x B]\!]$, and let's abbreviate $\phi + (t/x)$ by ϕ'. We also assume that x is not substituted by ϕ.
For the direct inclusion, suppose that $\vdash \Gamma, \phi' B(x)$ for any ground t. This is in particular valid for a fresh constant c, so we have a (cut free) proof of $\vdash \Gamma, \phi B(c)$ (where ϕ' has been replaced by ϕ since x is no more free in B. Replacing c by a fresh variable x does not changes anything to the proof. We can then apply the $\forall$-r rule:

$$\frac{\vdash \Gamma, \phi B(x)}{\vdash \Gamma \phi \forall x B(x)}$$

this proof justifies the fact that $\Gamma \in [\![\phi \forall x B(x)]\!]$.
Conversely, let Γ such that $\vdash \Gamma, \phi \forall x B(x)$ has a proof π. Let t be any ground term. We show that $\vdash \Gamma, \phi' B(x)$ has a proof, and that allows us to conclude. We construct as usual this proof by induction on π. When we reach the $\forall$-r rule on it, we have a proof π' of $\vdash \Gamma, \phi B(x)$. We can replace everywhere x by t in this proof, hence π'

also yields a proof of $\vdash \Gamma, \phi B(t)$, that is what we wanted.

Now, suppose A is negative. We must show $\bigcap_{t\text{ground}} \{\{\phi B(t)\}^{\perp\perp}\} = \{\phi\forall x B(x)\}^{\perp\perp} = \{\exists x B(x)^{\perp}\}^{\perp\perp}$.

For the direct inclusion, asusme that $\Gamma \in \{\phi B(t)\}^{\perp\perp}$ for any t. Thus, given a Δ such that $\vdash \Delta, \phi B(t)^{\perp}$ has a proof, we know that $\vdash \Delta, \Gamma$ has a proof. Now, let $\Sigma \in \{\exists x B(x)^{\perp}\}^{\perp}$, i.e. such that $\vdash \exists x B(x)^{\perp}, \Sigma$ has a proof π. We must show that the sequent $\vdash \Gamma, \Sigma$ has a proof to conclude. We construct this proof by induction on the structure of π, copying every rule unless we have the premise $\vdash B(t)^{\perp}, \Sigma$, where we use the hypothesis on Γ to exhibit a proof of $\vdash \Gamma, \Sigma$. This allows us to conclude. Conversely, let $\Gamma \in \{\exists x B(x)^{\perp}\}^{\perp\perp}$, i.e. such for any Δ such that sequent $\vdash \exists x B(x)^{\perp}, \Delta$ has a proof, the sequent $\vdash \Gamma, \Delta$ has also a proof. Now, let t be a ground term and $\Sigma \in \{B(t)^{\perp}\}^{\perp}$. The following inference step is valid:

$$\frac{\vdash B(t)^{\perp}, \Sigma}{\vdash \exists x B(x)^{\perp}, \Sigma}\ \exists\text{-r}$$

Hence, by hypothesis on Γ we have a proof of $\vdash \Gamma, \Sigma$. The conclusion follows directly.

- if $A = !B$. Suppose A is positive. We must show $[\![A]\!] = ![\![B]\!] = (\mathbf{1} \cap J \cap [\![B]\!])^{\perp\perp}$. For the direct inclusion, let Γ such that $\vdash \Gamma, !B$ has a proof π. Let $\Delta \in \mathbf{1} \cap J \cap [\![B]\!]^{\perp}$. We construct a proof of $\vdash \Gamma, \Delta$ by induction on π; copying every rule unless the last rule applied is promotion (this is the only rule applicable on $!B$). The we have a proof of the premise $\vdash \Gamma, B$ and we know that $\Gamma \equiv ?\Gamma'$. This shows that: $\Gamma \in [\![B]\!]$, $\Gamma \in J$ and $\Gamma \in \mathbf{1}$[1]. By hypothesis on Δ we have a proof of $\vdash \Gamma, \Delta$.

  Conversely, let $\Gamma \in (\mathbf{1} \cap J \cap [\![B]\!])^{\perp\perp}$. We must find a proof of $\vdash \Gamma, !B$. This follows directly from the fact that $!B \in (\mathbf{1} \cap J \cap [\![B]\!])^{\perp}$. Indeed, given a Δ such that $\vdash \Delta, B$ has a proof, $\Delta \equiv ?\Delta'$ and $\Delta \in \mathbf{1}$, the following rule can be applied:

$$\frac{\vdash \Delta, B}{\vdash \Delta, !B}\ \text{promotion}$$

  Suppose now that A is negative. We must show $\{?B^{\perp}\}^{\perp\perp} = (\mathbf{1} \cap J \cap \{B\}^{\perp\perp})^{\perp\perp}$ to be able to conclude, or equivalently $(\mathbf{1} \cap J \cap \{B\}^{\perp\perp})^{\perp} = \{?B^{\perp}\}^{\perp}$. For the direct inclusion, Let $\Gamma \in (\mathbf{1} \cap J \cap \{B\}^{\perp\perp})^{\perp}$. We must find a prof of $\vdash ?B^{\perp}, \Gamma$. It is sufficient to show that $?B^{\perp} \in \mathbf{1} \cap J \cap \{B\}^{\perp\perp}$. The only non trivial part is $?B^{\perp} \in \{B\}^{\perp\perp}$. Let $\Delta \in \{B^{\perp}\}^{\perp}$. The following inference:

$$\frac{\vdash \Delta, B^{\perp}}{\vdash \Delta, ?B^{\perp}}\ \text{dereliction}$$

  shows that this is really the case.

  Conversely, consider a context Γ such that $\vdash \Gamma, ?B^{\perp}$ has a proof π. Let $\Delta \in \mathbf{1} \cap J \cap \{B\}^{\perp\perp}$. Before initiating an induction on π we need to strengthen the hypothesis,

[1] if $\vdash \Sigma$ then $\vdash \Sigma, \Gamma$ by weakening so $\Gamma \in \perp^{\perp}$

and suppose that π is a proof of $\vdash \Gamma, ?B_n^\perp$ for some n, where $?B_n^\perp$ is an abbreviation for $?B_n^\perp$ repeated n times. This is indeed the case (with $n = 1$) by hypothesis.
We construct by induction on π a proof of $\vdash \Gamma, \Delta_n$, copying every rule unless it is a rule on $?B^\perp$ It could be:

- a dereliction. Then we have a proof of $\vdash \Gamma, B^\perp, ?B_{n-1}^\perp$. Hence $\Gamma \in \{B^\perp\}^\perp$. By induction hypothesis, we have a proof of $\vdash \Gamma, B^\perp, \Delta_{n-1}$, and since $\Delta \in \{B\}^{\perp\perp}$, it means that it can be turned into a proof of $\vdash \Gamma, \Delta, \Delta_{n-1}$.
- a weakening. We have a proof of $\vdash \Gamma, ?B_{n-1}^\perp$ as premise. We apply induction hypothesis, and obtain a proof of $\vdash \Gamma, \Delta_{n-1}$. We conclude using the fact that $\Delta \in \mathbf{1}$. Notice that this case also work if $n - 1 = 0$.
- a contraction. We have a proof of $\vdash \Gamma, ?B_{n+1}^\perp$. We apply induction hypothesis and obtain a proof of $\vdash \Gamma, \Delta_{n+1}$. Since $\Delta \in J$ we can contract Δ to obtain a proof of $\vdash, \Delta_n$.

Applying this result to the proof of $\vdash \Gamma, ?B^\perp$, we obtain a proof of $\vdash \Gamma, \Delta$. This concludes the proof.

– the other cases are the dual of the precedent ones. For instance, the positive case for $A = B \parr C$ uses exactly the same arguments that the negative case of the $\otimes$ connector. After using induction hypothesis on B and C, we must show:

$$(\llbracket B \rrbracket^\perp . \llbracket C \rrbracket^\perp)^\perp = \llbracket B \parr C \rrbracket$$

For the direct inclusion, let $\Gamma \in (\llbracket B \rrbracket^\perp . \llbracket C \rrbracket^\perp)^\perp$. Since $B \in \llbracket B \rrbracket^\perp$ and $C \in \llbracket C \rrbracket^\perp$, we have a proof of $\vdash \Gamma, B, C$. We conclude by applying $\parr$-r to this sequent, exactly as in the negative $\otimes$ case.
For the reverse inclusion, let Γ such that $\vdash \Gamma, B \parr C$. Let $\Delta_B \in \llbracket B \rrbracket^\perp$ and $\Delta_C \in \llbracket C \rrbracket^\perp$. We must find a proof of the sequent $\vdash \Gamma, \Delta_B, \Delta_C$. We construct it by induction on the proof of $\vdash \Gamma, B \parr C$. For the case of a $\parr$-r rule on $B \parr C$, we have a proof of a premise $\vdash \Gamma, B, C$. This shows that $\Gamma \cup \{B\} \in \llbracket C \rrbracket$ hence we have a proof of $\vdash \Gamma, B, \Delta_C$. And so on, as in the negative $\otimes$ case.

Lemma 8. *Let $A \equiv B$ two formulas. Then for any assignement ϕ we have $A_\phi^* = B_\phi^*$.*

Proof. We first focus on a single rewrite (propositional) rule: if $P \to A$, then, P and A are both positive or negative formulas. We conclude using Lem. 7.

We then focus on a single rewrite step. Consider two propositions $A \to^1 B$. We prove that $A_\phi^* = B_\phi^*$ by a straightforward induction over the proposition structure. The base case is the previous point.

Then, we extend it trivially to the reflexive-symmetric-transitive closure of the rewriting relation.

This lemma shows that the phase model we constructed is a model of $\mathcal{R}$. Hence positive rewrite systems admit cut elimination.

Conjecture 1. Wonder: can we generalize and classify two different ground instances of P one as positive and the other as negative ? Check that more carefully, possibly with the help of Richard.

We can now prove the main lemma in this case:

Lemma 9. *Let A be any proposition, and ϕ be an assignment. Let σ be the substitution associated to ϕ. Then:*

$$\sigma A^{\perp} \in A^*_{\phi} \subseteq [\![\sigma A]\!]$$

Proof. The proof of this lemma is completely independent from the proofs of Lem. 7 and 8. Its proof is standard, by induction over the formula structure, see for instance [8]. TODO HERE !!!

5.3 Mixing both conditions

We now consider a confluent and terminating rewrite system $\mathcal{R} = \mathcal{R}_{>} \cup \mathcal{R}_{+}$ such that the subsystem $\mathcal{R}_{>}$ verifies the order condition of Sec. 5.1 and the subsystem $\mathcal{R}_{+}$ verifies the positivity condition of Sec. 5.2. Moreover, $\mathcal{R}_{+}$ should be right normal for $\mathcal{R}$:

Definition 10 (Right normality). *Consider two rewrite systems $\mathcal{R}_1$ and $\mathcal{R}_2$.*

$\mathcal{R}_1$ is said right normal for $\mathcal{R}_2$ if, for any propositional rewrite rule $P \to A \in \mathcal{R}_1$, any instance of any atomic formulas occuring in A by a $\mathcal{R}_2$-normal substitution (associating normal terms for $\mathcal{R}_2$ to variables) is in normal form with respect to $\mathcal{R}_2$.

We construct the model as follows:

Definition 11. *We let S be the set of all ground terms of the language. Let f be a function symbol and P be a predicate symbol. We let:*

$$\hat{f} : (t_1, ..., t_n) \mapsto f(t_1, ..., t_n)$$

and

$$\hat{P} : (t_1, ..., t_n) \mapsto \begin{cases} \{P(t_1, ..., t_n)\}^{\perp\perp} & \text{if } P(t_1, ..., t_n) \text{ is in } \mathcal{R}_{>} \text{ normal form and negative} \\ [\![P(t_1, ..., t_n)]\!] & \text{if } P(t_1, ..., t_n) \text{ is in } \mathcal{R}_{>} \text{ normal form and positive} \\ (P(t_1, ..., t_n)\downarrow_{\mathcal{R}_{>}})^*_{\emptyset} & \text{otherwise, where } \emptyset \text{ is the empty assignement.} \end{cases}$$

The interpretation of compound formulas is defined at the same time, as in Def. 4.

This definition is well founded, since we do it by induction over $>$. As usual, we should now prove that this is a model of the rewrite system.

Lemma 10. *Let A be a formula, ϕ be an assignement. Let σ be the substitution associated to this assignement.*

1. $A^*_{\phi} = (\sigma A\downarrow_{\mathcal{R}_{>}})^*_{\phi}$.
2. *the model is a model of $\mathcal{R}_{>}$.*
3. $\sigma A^{\perp} \in A^*_{\phi} \subseteq [\![\sigma A]\!]$
4. *suppose that A is positive (resp. negative) and that any instance of any atomic formula occuring in A by $\mathcal{R}_{>}$-normal substitutions are normal for $\mathcal{R}_{>}$.*
   Then $A^*_{\phi} = [\![\sigma A]\!]$ (resp. $A^*_{\phi} = \{\sigma A\}^{\perp\perp}$.
5. *the model is a model of $\mathcal{R}$.*

Proof. 1. exactly the same proof as the proof of Lem. 4.
2. same proof as Lem. 5.

3. exactly the same proof as the proof of the main lemma Lem. 6. It uses the previous point.
   These first three points handles the $\mathcal{R}_>$ part of $\mathcal{R}$. It is then similar to the order section 5.1.
4. by induction on the formula structure as in Lem. 7. This holds since we can never apply a $\mathcal{R}_>$ rule (by the $\mathcal{R}_>$-right normality condition) on A, so this proof is independent.
5. the model is already a model of $\mathcal{R}_>$. Now, it is a consequence of the previous point that it is a model of $\mathcal{R}_+$. The proof is exactly the same as the proof of Lem. 8, in three points: atomic positive rewrite step, positive rewrite step and extension to the equivalence $\equiv$ modulo $\mathcal{R}$.

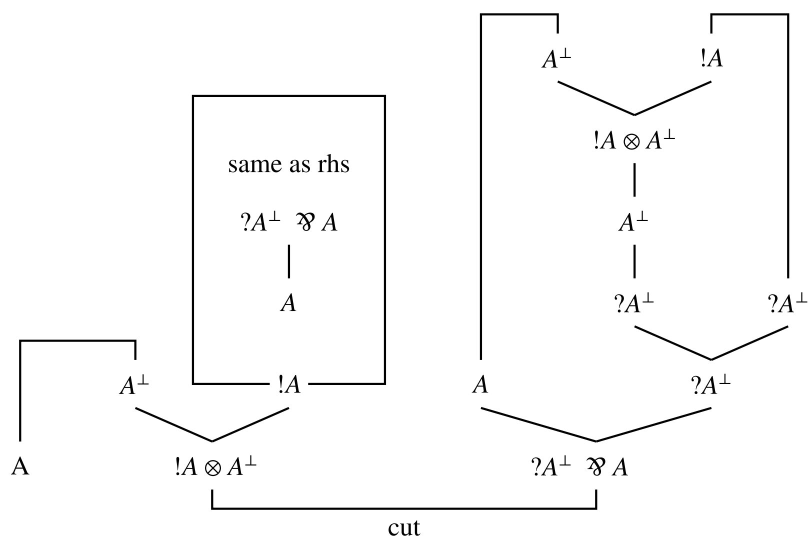
6 Proof Nets Modulo

They are basically the same as ordinary proof nets. However, the type of a node may change, since rewriting is allowed. Hence, more proof nets (and also, as we shall see, non normalizing ones) can be typed.

For an instance of that (and for those that already know what is a proof net), consider the rewrite system formed by the only rewrite rule:

$$A \rightarrow ?A^{\perp} \parr A$$

(which is to be understood as $(!A) \multimap A$), then we can type the following proof structure:



In this proof structure, apart from the usual links (see [4, 1] for instance), we have the rewrite link:

$$\begin{array}{l} \text{B} \\ \big| \,\text{rewrite, if } A \equiv B \\ \text{A} \end{array}$$

The introduction of this new link is not essential. We present things this way just to explicit the rewrite steps. This link could be forgotten in the sense that in a proof structure, we could collapse any rewrite link just replacing by its upper premise (the rewritted proposition) or by the lower premise (the proposition being rewritted), the former correponding to the usual presentation of deduction rules modulo ([2, 3]) as well as to the presentation done here, where the rewrite steps are done silently inside the rules.

The proof structure above does not normalizes. Indeed it reduces, after some few steps, to itself. On the other side, in our frame it is a valid proof net, and corresponds to the following proof:

$$
\dfrac{
  \otimes\text{-r}\;\dfrac{
    \text{promo.}\;\dfrac{
      \text{explicit rewrite}\;\dfrac{
        \dfrac{\dfrac{\vdots}{\text{as r.h.s.}}}{\vdash ?A^{\perp} \parr A}
      }{\vdash A}
    }{\vdash !A}
    \qquad A, A^{\perp}
  }{A, !A \otimes A^{\perp}}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{\vdash !A, ?A^{\perp} \qquad \vdash A, A^{\perp}}{\vdash A, !A \otimes A^{\perp}, ?A^{\perp}}\;\otimes\text{-r}
          }{\vdash A, A^{\perp}, ?A^{\perp}}\;\text{explicit rewrite}
        }{\vdash A, ?A^{\perp}, ?A^{\perp}}\;\text{dereliction}
      }{\vdash A, ?A^{\perp}}\;\text{contr.}
    }{\vdash ?A^{\perp} \parr A}\;\parr\text{-r}
  }{\vdash A}\;\text{cut}
$$

Indeed, the sequentialization and correctness criterion are the same, save for the addition rewrite link, which has to be erased before sequentialization.

7 A proof term assignment system

See what is done in University Paris XIII for instance.

8 Focusing

9 Examples

10 Conclusion

We believe this work can be shifted in higher-order logic as well as in intuitionistic linear logic. In the first case, we will have to stregthen some lemma, because we will encounter free proposition variables, hence lemmas as Lem. 4 should be formulated in an other manner. In the later case, we should use the weakened version of phase semantics of [9].

We would like to replace the connector ! by $!A \rightarrow !A \otimes A$ and $!A \rightarrow \mathbf{1}$

References

1. V. Danos and L. Regnier. The structure of multiplicatives. *Archive for Mathematical Logic*, 28:181–203, 1989.

2. G. Dowek, T. Hardin, and C. Kirchner. Theorem proving modulo. *Journal of Automated Reasoning*, 31:33–72, 2003.
3. G. Dowek and B. Werner. Proof normalization modulo. *The Journal of Symbolic Logic*, 68 (4):1289–1316, December 2003.
4. J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
5. O. Hermant. A model-based cut elimination proof. *2nd St-Petersburg Days of Logic and Computability*, 2003.
6. O. Hermant. *Méthodes Sémantiques en Déduction Modulo*. PhD thesis, Université Paris 7 – Denis Diderot, 2005.
7. O. Hermant. Semantic cut elimination in the intuitionistic sequent calculus. In P. Urzyczyn, editor, *Typed Lambda-Calculi and Applications*, volume 3461 of *LNCS*, pages 221–233. Springer, Nara, Japan, 2005.
8. M. Okada. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science*, 227:333–396, 1999.
9. M. Okada. A uniform semantic proof for cut-elimination and completeness of various first and higher order logics. *Theoretical Computer Science*, 281:471–498, 2002.