Deduction Modulo

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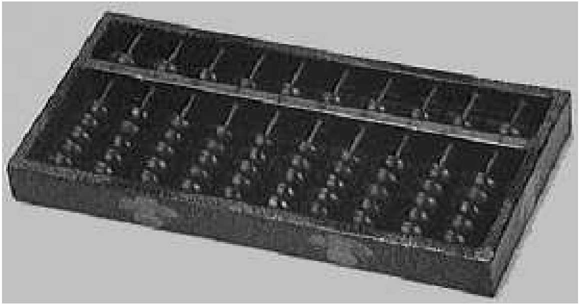
Deduction and Computation



Deduction system: Gentzen's sequent calculus

$$\frac{}{\Gamma, P \vdash P}\text{axiom} \qquad \frac{\Gamma, P \vdash Q \quad \Gamma \vdash P}{\Gamma \vdash Q}\text{cut}$$

$$\frac{\Gamma, P, P \vdash Q}{\Gamma, P \vdash Q}\text{contr-l} \qquad \frac{}{\Gamma, \bot \vdash Q}\bot\text{-g}$$

$$\frac{\Gamma, P \vdash R \quad \Gamma, Q \vdash R}{\Gamma, P \vee Q \vdash R}\vee\text{-g} \qquad \frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\vee\text{-d} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\vee\text{-d}$$

$$\frac{\Gamma \vdash P \quad \Gamma, Q \vdash R}{\Gamma, P \Rightarrow Q \vdash R}\Rightarrow\text{-g} \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \Rightarrow Q}\Rightarrow\text{-d}$$

$$\frac{\Gamma, \{c/x\}P \vdash Q}{\Gamma, \exists x P \vdash Q}\exists\text{-g}, \ c \text{ fresh} \qquad \frac{\Gamma \vdash \{t/x\}P}{\Gamma \vdash \exists x P}\exists\text{-d}$$

The cut rule: a detour

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- we prove $\Gamma \vdash P$
- we assume P and prove $\Gamma, P \vdash Q$
- it is a proof of $\Gamma \vdash Q$

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- lemma application.

Deduction system: sequent calculus

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Axioms vs. rewriting

Axioms	Rewriting
$x + S(y) = S(x + y)$ $x + 0 = x$ $x * 0 = 0$ $x * S(y) = x + x * y$ $(x * y = 0) \Leftrightarrow (x = 0 \vee y = 0)$	$x + S(y) \rightarrow S(x + y)$ $x + 0 \rightarrow x$ $x * 0 \rightarrow 0$ $x * S(y) \rightarrow x + x * y$ $(x * y = 0) \rightarrow (x = 0 \vee y = 0)$
$\dfrac{\dfrac{\vdots}{\mathcal{T} \vdash 2 * 2 = 4}}{\mathcal{T} \vdash \exists x(2 * x = 4)}$	$\dfrac{\overline{\vdash_{\mathcal{R}} 4 = 4}}{\vdash_{\mathcal{R}} \exists x(2 * x = 4)}$

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- and on **propositions** :

$$x * y = 0 \longrightarrow x = 0 \vee y = 0$$

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Sequent calculus modulo

$$\frac{}{\Gamma, P \vdash Q}\text{axiom } P \equiv_{\mathcal{R}} Q \qquad \frac{\Gamma, P \vdash R \quad \Gamma \vdash Q}{\Gamma \vdash R}\text{cut } P \equiv_{\mathcal{R}} Q$$

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$$\frac{\Gamma \vdash P \quad \Gamma, Q \vdash R}{\Gamma, S \vdash R} \Rightarrow\text{-g } P \Rightarrow Q \equiv_{\mathcal{R}} S \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash S} \Rightarrow\text{-d } P \Rightarrow Q \equiv_{\mathcal{R}} S$$

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An example of rewriting theory: Peano/Heyting Arithmetic

As an axiomatic theory:

$$\forall(x)\forall(y)(S(x) = S(y) \Rightarrow x = y)$$

$$\forall x \neg(0 = S(x))$$

$$\{0/x\}P \Rightarrow \forall y(\{y/x\}P \Rightarrow \{S(y)/x\}P) \Rightarrow \forall n\{n/x\}P$$

$$\forall y(O + y = y) \qquad \forall x\forall y(S(x) + y = S(x + y))$$

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Orienting the last four equations is not hard:

$$0 + y \longrightarrow y \qquad S(x) + y \longrightarrow S(x + y)$$

$$0 \times y \longrightarrow 0 \qquad S(x) \times y \longrightarrow x \times y + y$$

Adding symbols

We define:

- a symbol *Pred* (for predecessor) and the axioms:

$$Pred(0) = 0 \qquad Pred(S(x)) = x$$

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- two predicate symbols N and $Null$, and the axioms:

$$N(0) \qquad \forall x (N(x) \Rightarrow N(S(x)))$$

$$Null(0) \qquad \forall x (\neg Null(S(x)))$$

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it is a conservative extension over PA/HA, up to a formulas traduction:

$$|\forall x P| = \forall x (N(x) \Rightarrow P)$$

Handling equality and induction

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- Why all this ?

Arithmetic reformulated

$$\forall y \forall z (y = z \Leftrightarrow \forall p (y \in p \Rightarrow z \in p))$$
$$\forall n (N(n) \Leftrightarrow \forall p (0 \in p \Rightarrow \forall y (N(y) \Rightarrow y \in p \Rightarrow S(y) \in p) \Rightarrow n \in p))$$
$$\forall x \forall y_1 \ldots \forall y_n (x \in f_{x, y_1, \ldots, y_n, P}(y_1, \ldots, y_n) \Leftrightarrow P)$$

$$Pred(0) = 0 \qquad \forall x (Pred(S(x)) = x)$$
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$$\forall y (0 \times y = 0) \qquad \forall x \forall y (S(x) \times y = x \times y + y)$$

This formulation is conservative over PA

Arithmetic modulo

$$y = z \longrightarrow \forall p (y \in p \Rightarrow z \in p)$$

$$N(n) \longrightarrow \forall p (0 \in p \Rightarrow \forall y (N(y) \Rightarrow y \in p \Rightarrow S(y) \in p) \Rightarrow n \in p)$$

$$x \in f_{x,y_1,\ldots,y_n,P}(y_1, \ldots, y_n) \longrightarrow P$$

$$Pred(0) \longrightarrow 0 \qquad Pred(S(x)) \longrightarrow x$$

$$Null(0) \longrightarrow \top \qquad Null(S(x)) \longrightarrow \bot$$

$$0 + y \longrightarrow y \qquad S(x) + y \longrightarrow S(x + y)$$

$$0 \times y \longrightarrow 0 \qquad S(x) \times y \longrightarrow x \times y + y$$

This forms a rewrite system $\mathcal{R}_{HA}$

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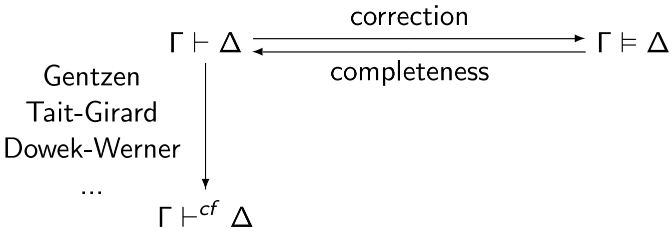
- a cut in deduction modulo corresponds to *ad hoc* axiomatic cuts of axiomatic theories.

How to eliminate cut



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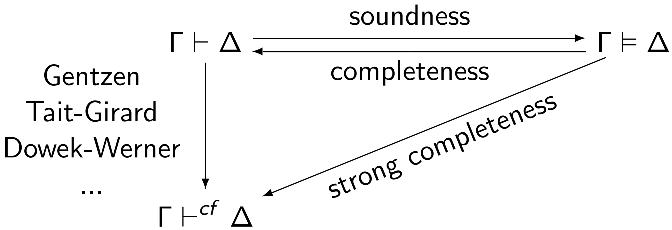

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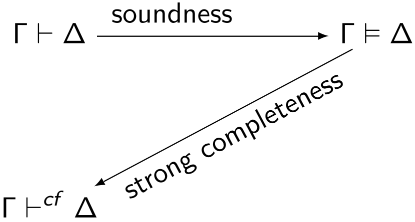

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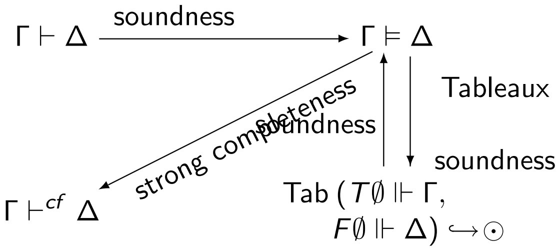

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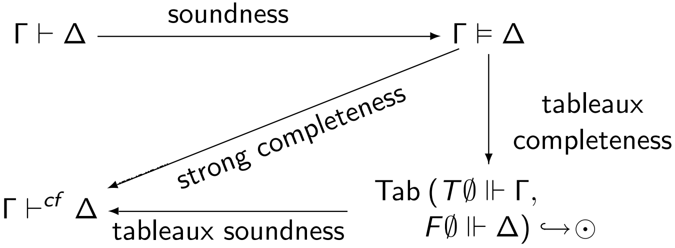

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- we construct Hintikka sets/ complete tableaux.
- we have to go further: the obtained Hintikka set has to be transformed into a model of $\mathcal{R}$ (most tedious part).

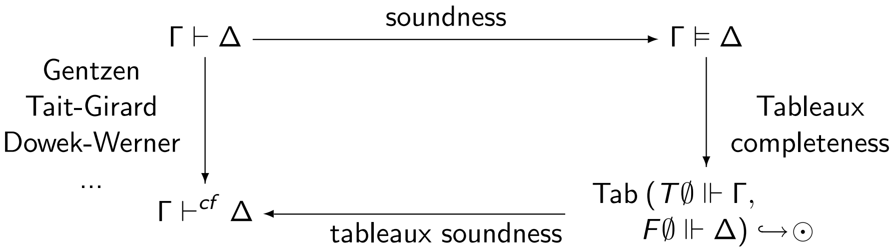
Results with the semantic method

Cut elimination for:

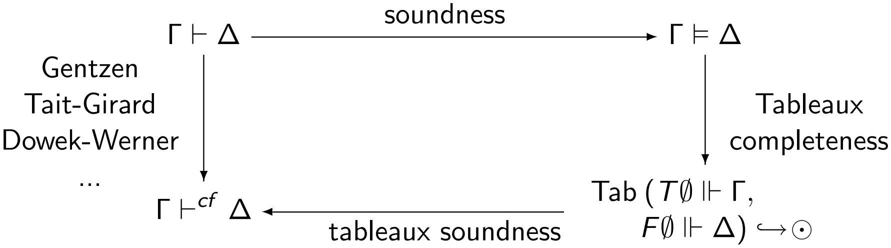
- a w.f.o. condition on $\mathcal{R}$
- a positivity condition on $\mathcal{R}$
- a mix of the two previous conditions
- HOL formulation in Deduction Modulo
- the rule:

$$R \in R \rightarrow \forall y(y \simeq R \Rightarrow (y \in R \Rightarrow (A \Rightarrow \neg A)))$$

does not have proof normalization, but has cut admissibility.



- both approach are not equivalent.



- both approach are not equivalent.
- this is still a field of investigations.