Semantic Cut Elimination in Sequent Calculus

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- We will prove cut elimination:
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  - by semantic means, extending results for classical logic
  - and extend it to deduction modulo
- Is there a link with normalisation method ?

The Intuitionnistic Sequent Calculus

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$$\frac{\Gamma, \{c/x\}P \vdash Q}{\Gamma, \exists x P \vdash Q}\exists\text{-l, } c \text{ fresh constant} \qquad \frac{\Gamma \vdash \{t/x\}P}{\Gamma \vdash \exists x P}\exists\text{-r}$$

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  - Kripke Structures

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- $\Vdash$ is a relation between worlds and propositions, verifying certain condition, among which:
  - A is an atom: if $\alpha \leq \beta$ and $\alpha \Vdash A$, then $\beta \Vdash A$.
  - $\alpha \Vdash P \Rightarrow Q$ iff for any $\beta \geq \alpha$ we have $\beta \Vdash P$ implies $\beta \Vdash Q$.

- Another formulation of the completeness theorem: if $\Gamma \not\vdash^{cf} P$ there exists a K.S. that is a model for Γ and that is not a model for P.
  Given such Γ qnd P, we have to construct a world α of some KS $\mathcal{K}$, s.t. $\alpha \Vdash \Gamma$ and $\alpha \not\Vdash P$.

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- We can no more use usual definitions (e.g. completeness or saturation), because we can't no more ensure that $\Gamma \vdash^{cf} P$ implies $P \in \Gamma$, without having inconsitency.
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- Adding the fact that we are in an intuitionnistic framework, they become:
  - A-Consistency: $\Gamma \not\vdash^{cf} A$
  - A-Completeness (saturation): $\Gamma, P \vdash^{cf} A$ or $P \in \Gamma$
  - A-Henkin witnesses: $\Gamma, \exists x P \not\vdash^{cf} A$ implies $\{c/x\}P \in \Gamma$ for some constant c.

Model Construction

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- This **is** a Kripke Structure. Moreover, it has the following properties:
  - for any $P \in \Gamma$, $\Gamma \Vdash P$
  - if $\Gamma \nvdash^{cf} P$ then $\Gamma \nVdash P$

Model Construction

- Turn back to the proof of the completeness theorem.
  Theorem: if $T \not\Vdash A$, then we can find $\alpha \Vdash T$ and $\alpha \not\Vdash A$.
  Proof: We complete T into Γ, and in the KS previously defined, we get that $\Gamma \Vdash T$. $\square$

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- As a corollary, we get the cut elimination theorem.
  Theorem: if $\Gamma \vdash P$ then $\Gamma \vdash^{cf} P$.
  Proof: We have $\Gamma \Vdash P$ by soundness, and we conclude with cut-free completeness.

The Deduction Modulo

- Addition of rewrite rules on terms:

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(x+y)+z & \longrightarrow & x+(y+z) \\
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- Rewrite rules on propositions is the central paradigm of deduction modulo.

The Deduction Modulo

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Reflexivity}}{\mathcal{P} \vdash 4 = 4}}{\vdots}}{\mathcal{P} \vdash 3 + 1 = 4}}{\dfrac{\vdots}{\mathcal{P} \vdash 2 + 2 = 4}}$$

Replacing axiom with rewrite rule $x + S(y) \rightarrow S(x) + y$:

$$\dfrac{\text{Reflexivity}}{\vdash_{\mathcal{R}} 2 + 2 = 4}$$

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  - cut-free A-Henkin witnesses: $\Gamma, \exists x Q \nvdash^{cf}_{\mathcal{R}} A$ implies $\{c/x\}Q \in \Gamma$.

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- The previous method should then be modified, because we have to ensure this new condition.

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- The completion process remains the same, but uses the new definitions.
- The model definition depends on the class of the rewrite rules. We show it for one class of Rewrite Systems.

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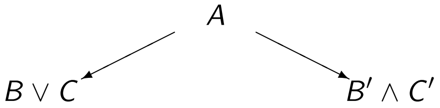
- Confluence is necessary, in order not to have:



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- We extend it by induction on the order $\prec$ to non atomic propositions and non-normal atoms:
  - for a non-normal atom, set $\Gamma \Vdash A$ iff $\Gamma \Vdash A \downarrow$
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- From the well-foundedness of $\prec$, the above definition is well-founded.

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- As a corollary, we get the cut-elimination theorem:
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$$\Gamma \vdash_{\mathcal{R}} P \quad \text{implies } \Gamma \vdash_{\mathcal{R}}^{cf} P$$

Cut elimination $\neq$ Normalization

The rewrite system composed of the following rewrite rule:

$$R \in R \rightarrow_{\mathcal{R}} \forall y\ (\forall x(y \in x \Rightarrow R \in x) \Rightarrow (y \in R \Rightarrow (A \Rightarrow A))$$

- Doesn't normalize. E.g. the following proof:

$$\frac{R \in R \vdash^{cf}_{\mathcal{R}} A \Rightarrow A \quad \vdash^{cf}_{\mathcal{R}} R \in R}{\vdash_{\mathcal{R}} A \Rightarrow A}\ \text{cut}$$

- Has the cut elimination property. We can find by other means a cut-free proof of $\vdash^{cf}_{\mathcal{R}} A \Rightarrow A$.
- And a proof of cut redundancy by our method works.

Further work

- Can we extend these results to other classes (HOL, positive Rewrite Systems, ...)
- Links with methods based on Heyting Algebras ?