Encoding Zenon Modulo in Dedukti

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Double-Negation Translations

Double-Negation translations:

- a shallow way to encode classical logic into intuitionistic
- Zenon modulo's backend for Dedukti
- existing translations: Kolmogorov's (1925), Gentzen-Gödel's (1933), Kuroda's (1951), Krivine's (1990), $\cdots$

Minimizing the translations:

- turns more formulæ into themselves;
- shifts a classical proof into an intuitionistic proof of the *same* formula.

- in this talk first-order logic (no modulo)
- readily extensible

The Classical Sequent Calculus (LK)

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x], \Delta}{\Gamma \vdash \forall x A, \Delta}\ \forall_R$$

The Intuitionistic Sequent Calculus (LJ)

$$\frac{}{\Gamma, A \vdash A}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{R2}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x]}{\Gamma \vdash \forall x A}\ \forall_R$$

Note on Frameworks

- structural rules are not shown (contraction, weakening)
- left-rules seem very similar in both cases
- so, lhs formulæ can be translated by themselves
- this accounts for polarizing the translations

Positive and Negative occurrences

- An occurrence of A in B is positive if:
  - $B = A$
  - B = $C \star D$ [$\star = \wedge, \vee$] and the occurrence of A is in C or in D and positive
  - B = $C \Rightarrow D$ and the occurrence of A is in C (resp. in D) and negative (resp. positive)
  - B = $Qx\ C$ [$Q = \forall, \exists$] and the occurrence of A is in C and is positive
- Dually for negative occurrences.

Kolmogorov's Translation

Kolmogorov's ¬¬-translation introduces ¬¬ everywhere:

$$\begin{array}{rcll}
B^{Ko} & = & \neg\neg B & \text{(atoms)} \\
(B \wedge C)^{Ko} & = & \neg\neg(B^{Ko} \wedge C^{Ko}) & \\
(B \vee C)^{Ko} & = & \neg\neg(B^{Ko} \vee C^{Ko}) & \\
(B \Rightarrow C)^{Ko} & = & \neg\neg(B^{Ko} \Rightarrow C^{Ko}) & \\
(\forall x A)^{Ko} & = & \neg\neg(\forall x A^{Ko}) & \\
(\exists x A)^{Ko} & = & \neg\neg(\exists x A^{Ko}) &
\end{array}$$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{Ko}, \lrcorner\Delta^{Ko} \vdash$ is provable in LJ.

Antinegation

$\lrcorner$ is an operator, such that:

- $\lrcorner\neg A = A$;
- $\lrcorner B = \neg B$ otherwise.

Light Kolmogorov's Translation

Moving negation from connectives to formulæ [DowekWerner]:

$$
\begin{array}{rcll}
B^K & = & B & \text{(atoms)} \\
(B \wedge C)^K & = & (\neg\neg B^K \wedge \neg\neg C^K) & \\
(B \vee C)^K & = & (\neg\neg B^K \vee \neg\neg C^K) & \\
(B \Rightarrow C)^K & = & (\neg\neg B^K \Rightarrow \neg\neg C^K) & \\
(\forall x A)^K & = & \forall x \neg\neg A^K & \\
(\exists x A)^K & = & \exists x \neg\neg A^K &
\end{array}
$$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^K, \neg\Delta^K \vdash$ is provable in LJ.

Correspondence

$A^{Ko} = \neg\neg A^K$

Polarizing Kolmogorov's translation

Warming-up. Consider left-hand and right-hand side formulæ:

LHS

$$
\begin{aligned}
B^K &= B \\
(B \wedge C)^K &= (\neg\neg B^K \wedge \neg\neg C^K) \\
(B \vee C)^K &= (\neg\neg B^K \vee \neg\neg C^K) \\
(B \Rightarrow C)^K &= (\neg\neg B^K \Rightarrow \neg\neg C^K) \\
(\forall x A)^K &= \forall x \neg\neg A^K \\
(\exists x A)^K &= \exists x \neg\neg A^K
\end{aligned}
$$

RHS

$$
\begin{aligned}
B^K &= B \\
(B \wedge C)^K &= (\neg\neg B^K \wedge \neg\neg C^K) \\
(B \vee C)^K &= (\neg\neg B^K \vee \neg\neg C^K) \\
(B \Rightarrow C)^K &= (\neg\neg B^K \Rightarrow \neg\neg C^K) \\
(\forall x A)^K &= \forall x \neg\neg A^K \\
(\exists x A)^K &= \exists x \neg\neg A^K
\end{aligned}
$$

Example of translation

$((A \vee B) \Rightarrow C)^K$ is $\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg C$

$((A \vee B) \Rightarrow C)^K$ is $\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg C$

Polarizing Light Kolmogorov's Translation

Warming-up. Consider left-hand and right-hand side formulæ:

LHS

$$
\begin{aligned}
B^{K+} &= B \\
(B \wedge C)^{K+} &= (B^{K+} \wedge C^{K+}) \\
(B \vee C)^{K+} &= (B^{K+} \vee C^{K+}) \\
(B \Rightarrow C)^{K+} &= (\neg\neg B^{K-} \Rightarrow C^{K+}) \\
(\forall x A)^{K+} &= \forall x A^{K+} \\
(\exists x A)^{K+} &= \exists x A^{K+}
\end{aligned}
$$

RHS

$$
\begin{aligned}
B^{K-} &= B \\
(B \wedge C)^{K-} &= (\neg\neg B^{K-} \wedge \neg\neg C^{K-}) \\
(B \vee C)^{K-} &= (\neg\neg B^{K-} \vee \neg\neg C^{K-}) \\
(B \Rightarrow C)^{K-} &= (B^{K+} \Rightarrow \neg\neg C^{K-}) \\
(\forall x A)^{K-} &= \forall x \neg\neg A^{K-} \\
(\exists x A)^{K-} &= \exists x \neg\neg A^{K-}
\end{aligned}
$$

Example of translation

$((A \vee B) \Rightarrow C)^{K+}$ is $\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow C$

$((A \vee B) \Rightarrow C)^{K-}$ is $(A \vee B) \Rightarrow \neg\neg C$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is bouncing. Example:

$$\wedge_R \frac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\frac{\dfrac{\pi'_1}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash} \quad \dfrac{\pi'_2}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash} \wedge_R$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is bouncing. Example:

$$\wedge_R \; \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_L \; \dfrac{\dfrac{\dfrac{\pi_1'}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash} \qquad \dfrac{\pi_2'}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}} \wedge_R}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is bouncing. Example:

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_L \dfrac{\wedge_R \dfrac{\dfrac{\dfrac{\pi'_1}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-}} \quad \dfrac{\dfrac{\pi'_2}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg B^{K-}}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}}}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is bouncing. Example:

$$\wedge_R \ \frac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_L \ \frac{\wedge_R \ \dfrac{\neg_R \ \dfrac{\dfrac{\pi'_1}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-}} \qquad \neg_R \ \dfrac{\dfrac{\pi'_2}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg B^{K-}}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}}}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is bouncing. Example:

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L\dfrac{\dfrac{\neg_R\dfrac{\dfrac{\pi'_1}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-}} \qquad \neg_R\dfrac{\dfrac{\pi'_2}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg B^{K-}}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}} \wedge_R}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}$$

Theorem

If $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ, then $\Gamma \vdash \Delta$ is provable in LK.

Proof: ad-hoc generalization.

Gödel-Gentzen Translation

In this translation, disjunctions and existential quantifiers are replaced by a combination of negation and their De Morgan duals:

LHS	RHS
$B^{gg} = \neg\neg B$	$B^{gg} = \neg\neg B$
$(A \wedge B)^{gg} = A^{gg} \wedge B^{gg}$	$(A \wedge B)^{gg} = A^{gg} \wedge B^{gg}$
$(A \vee B)^{gg} = \neg(\neg A^{gg} \wedge \neg B^{gg})$	$(A \vee B)^{gg} = \neg(\neg A^{gg} \wedge \neg B^{gg})$
$(A \Rightarrow B)^{gg} = A^{gg} \Rightarrow B^{gg}$	$(A \Rightarrow B)^{gg} = A^{gg} \Rightarrow B^{gg}$
$(\forall x A)^{gg} = \forall x A^{gg}$	$(\forall x A)^{gg} = \forall x A^{gg}$
$(\exists x A)^{gg} = \neg\forall x \neg A^{gg}$	$(\exists x A)^{gg} = \neg\forall x \neg A^{gg}$

Example of translation

$((A \vee B) \Rightarrow C)^{gg}$ is $(\neg(\neg\neg\neg A \wedge \neg\neg\neg B)) \Rightarrow \neg\neg C$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{gg}, \lrcorner\Delta^{gg} \vdash$ is provable in LJ.

Polarizing Gödel-Gentzen translation

Let us apply the same idea on this translation:

LHS	RHS
$B^p = B$	$B^n = \neg\neg B$
$(B \wedge C)^p = B^p \wedge C^p$	$(B \wedge C)^n = B^n \wedge C^n$
$(B \vee C)^p = B^p \vee C^p$	$(B \vee C)^n = \neg(\neg B^n \wedge \neg C^n)$
$(B \Rightarrow C)^p = B^n \Rightarrow C^p$	$(B \Rightarrow C)^n = B^p \Rightarrow C^n$
$(\forall x B)^p = \forall x B^p$	$(\forall x B)^n = \forall x B^n$
$(\exists x B)^p = \exists x B^p$	$(\exists x B)^n = \neg\forall x \neg B^n$

Example of translation

$((A \vee B) \Rightarrow C)^p$ is $(\neg(\neg\neg\neg A \wedge \neg\neg\neg B)) \Rightarrow C$
$((A \vee B) \Rightarrow C)^n$ is $((A \vee B) \Rightarrow \neg\neg C$

Theorem ?

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{gg}, \lrcorner\Delta^{gg} \vdash$ is provable in LJ.

A Focus on LK $\rightarrow$ LJ

- less negations imposes more discipline. Example:

$$\wedge_R \frac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L \dfrac{\wedge_R \dfrac{?? \dfrac{\dfrac{\pi'_1}{\Gamma^p, \lrcorner A^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n} \qquad \dfrac{\dfrac{\pi'_2}{\Gamma^p, \lrcorner B^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash B^n} ??}{\Gamma^p, \lrcorner\Delta^n \vdash A^n \wedge B^n}}{\Gamma^p, \neg(A^n \wedge B^n), \lrcorner\Delta^n \vdash}$$

- when A^n introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^n$
- otherwise A^n remains of the rhs in the LJ proof.

A Focus on LK $\rightarrow$ LJ

- less negations imposes more discipline. Example:

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L \dfrac{\wedge_R \dfrac{?? \dfrac{\dfrac{\pi_1'}{\Gamma^p, \lrcorner A^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n} \quad ?? \dfrac{\dfrac{\pi_2'}{\Gamma^p, \lrcorner B^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash B^n}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n \wedge B^n}}{\Gamma^p, \neg(A^n \wedge B^n), \lrcorner\Delta^n \vdash}$$

- when A^n introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^n$
- otherwise A^n remains of the rhs in the LJ proof.
- the next rule in π_1 and π_2 must be on A (resp. B). How ?

A Focus on LK → LJ

- less negations imposes more discipline. Example:

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L \dfrac{\wedge_R \dfrac{?? \dfrac{\dfrac{\pi'_1}{\Gamma^p, \lrcorner A^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n} \quad ?? \dfrac{\dfrac{\pi'_2}{\Gamma^p, \lrcorner B^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash B^n}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n \wedge B^n}}{\Gamma^p, \neg(A^n \wedge B^n), \lrcorner\Delta^n \vdash}$$

- when A^n introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^n$
- otherwise A^n remains of the rhs in the LJ proof.
- the next rule in π_1 and π_2 must be on A (resp. B). How ?
- use Kleene's inversion lemma
- or ... this is exactly what focusing is about !

A Focused Classical Sequent Calculus

Sequent with focus

A focused sequent $\Gamma \vdash A; \Delta$ has three parts:

- Γ and Δ
- A, the (possibly empty) stoup formula

$$\Gamma \vdash \underbrace{\quad . \quad}_{\text{stoup}} ; \Delta$$

- when the stoup is not empty, the next rule must apply on its formula,
- under some conditions, it is possible to move/remove a formula in/from the stoup.

A Focused Sequent Calculus

$$\frac{}{\Gamma, A \vdash \ . \ ; A, \Delta}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \ . \ ; \Delta}{\Gamma, A \wedge B \vdash \ . \ ; \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A \ ; \Delta \qquad \Gamma \vdash B \ ; \Delta}{\Gamma \vdash A \wedge B \ ; \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \ . \ ; \Delta \qquad \Gamma, B \vdash \ . \ ; \Delta}{\Gamma, A \vee B \vdash \ . \ ; \Delta}\ \vee_L \qquad \frac{\Gamma \vdash \ . \ ; A, B, \Delta}{\Gamma \vdash \ . \ ; A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A \ ; \Delta \qquad \Gamma, B \vdash \ . \ ; \Delta}{\Gamma, A \Rightarrow B \vdash \ . \ ; \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B \ ; \Delta}{\Gamma \vdash A \Rightarrow B \ ; \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma, A[c/x] \vdash \ . \ ; \Delta}{\Gamma, \exists x A \vdash \ . \ ; \Delta}\ \exists_L \qquad \frac{\Gamma \vdash \ . \ ; A[t/x], \Delta}{\Gamma \vdash \ . \ ; \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \ . \ ; \Delta}{\Gamma, \forall x A \vdash \ . \ ; \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x] \ ; \Delta}{\Gamma \vdash \forall x A \ ; \Delta}\ \forall_R$$

$$\frac{\Gamma \vdash A \ ; \Delta}{\Gamma \vdash \ . \ ; A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash \ . \ ; A, \Delta}{\Gamma \vdash A \ ; \Delta}\ \text{release}$$

A Focused Sequent Calculus

$$\frac{\Gamma \vdash A \,;\, \Delta}{\Gamma \vdash \,.\,;\, A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash \,.\,;\, A, \Delta}{\Gamma \vdash A \,;\, \Delta}\ \text{release}$$

Characteristics:

- in release, A is either atomic or of the form $\exists x B$, $B \vee C$ or $\neg B$;
- in focus, the converse holds: A must not be atomic, nor of the form $\exists x B$, $B \vee C$ nor $\neg B$.
- the *synchronous* (outside the stoup) right-rules are $\exists_R$, $\neg_R$, $\vee_R$ and (atomic) axiom: the exact places where $\{.\}^n$ introduces negation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK then $\Gamma \vdash .; \Delta$ is provable.

Proof: use Kleene's inversion lemma (holds for all connectives/quantifiers, except $\exists_R$ and $\forall_L$).

Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash A\,;\,\Delta}{\Gamma \vdash .\,;\,A, \Delta}\text{focus} \qquad \frac{\Gamma \vdash .\,;\,A, \Delta}{\Gamma \vdash A\,;\,\Delta}\text{release}$$

Theorem

If $\Gamma \vdash A; \Delta$ in focused LK, then $\Gamma^p, \lrcorner\Delta^n \vdash A^n$ in LJ

- release is translated by the $\neg_R$ rule
- focus is translated by the $\neg_L$ rule

Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash A \,;\, \Delta}{\Gamma \vdash \,.\,;\, A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash \,.\,;\, A, \Delta}{\Gamma \vdash A \,;\, \Delta}\ \text{release}$$

Theorem

If $\Gamma \vdash A; \Delta$ in focused LK, then $\Gamma^p, \lrcorner\Delta^n \vdash A^n$ in LJ

- release is translated by the $\neg_R$ rule
- focus is translated by the $\neg_L$ rule
- $\lrcorner\Delta^n$ removes the trailing negation on $\exists^n$ $(\neg\forall\neg)$, $\vee^n$ $(\neg \wedge \neg)$, $\neg^n$ $(\neg)$ and atoms $(\neg\neg)$
- what a surprise: focus is forbidden on them, so rule on the lhs:

LK rule	$\exists_R$	$\neg_R$	$\vee_R$	ax.
LJ rule	$\forall_L$	nop	$\wedge_L$	$\neg_L$ + ax.

Going further: Kuroda's translation

Originating from Glivenko's remark for propositional logic:

Theorem[Glivenko]

if $\vdash A$ in LK, then $\vdash \neg\neg A$ in LJ.

Kuroda's $\neg\neg$-translation:

$$
\begin{array}{rcll}
B^{Ku} & = & B & \text{(atoms)} \\
(B \wedge C)^{Ku} & = & B^{Ku} \wedge C^{Ku} & \\
(B \vee C)^{Ku} & = & B^{Ku} \vee C^{Ku} & \\
(B \Rightarrow C)^{Ku} & = & B^{Ku} \Rightarrow C^{Ku} & \\
(\forall x A)^{Ku} & = & \neg\neg(\forall x A^{Ku}) & \\
(\exists x A)^{Ku} & = & \exists x A^{Ku} &
\end{array}
$$

Theorem[Kuroda]

$\Gamma \vdash \Delta$ in LK iff $\Gamma^{Ku}, \neg\Delta^{Ku} \vdash$ in LJ.

- restarts double-negation everytime we pass a universal quantifier.

Combining Kuroda's and Gentzen-Gödel's translations

- work of Frédéric Gilbert (2013), who noticed:
  1. Kuroda's translation of $\forall x \forall y A$

$$\forall x \neg\neg \forall y \neg\neg A \text{ can be simplified: } \forall x \forall y \neg\neg A$$

  2. $\neg\neg A$ itself can be treated *à la* Gentzen-Gödel
  3. and of course with polarization

Reminder:

Gödel-Gentzen	Kuroda
$\varphi(P) = \neg\neg P$	$\psi(P) = P$
$\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B)$	$\psi(A \wedge B) = \psi(A) \wedge \psi(B)$
$\varphi(A \vee B) = \neg\neg(\varphi(A) \vee \varphi(B))$	$\psi(A \vee B) = \psi(A) \vee \psi(B)$
$\varphi(A \Rightarrow B) = \varphi(A) \Rightarrow \varphi(B)$	$\psi(A \Rightarrow B) = \psi(A) \Rightarrow \psi(B)$
$\varphi(\exists x A) = \neg\neg \exists x \varphi(A)$	$\psi(\exists x A) = \exists x \psi(A)$
$\varphi(\forall x A) = \forall x \varphi(A)$	$\psi(\forall x A) = \forall x \neg\neg \psi(A)$

Combining Kuroda's and Gentzen-Gödel's translations

- How does it work ?

GG

$$\begin{aligned}
\varphi(P) &= \neg\neg P\\
\varphi(A \wedge B) &= \varphi(A) \wedge \varphi(B)\\
\varphi(A \vee B) &= \neg\neg(\varphi(A) \vee \varphi(B))\\
\varphi(A \Rightarrow B) &= \varphi(A) \Rightarrow \varphi(B)\\
\varphi(\exists x A) &= \neg\neg\exists x \varphi(A)\\
\varphi(\forall x A) &= \forall x \varphi(A)
\end{aligned}$$

Kuroda

$$\begin{aligned}
\psi(P) &= P\\
\psi(A \wedge B) &= \psi(A) \wedge \psi(B)\\
\psi(A \vee B) &= \psi(A) \vee \psi(B)\\
\psi(A \Rightarrow B) &= \psi(A) \Rightarrow \psi(B)\\
\psi(\exists x A) &= \exists x \psi(A)\\
\psi(\forall x A) &= \forall x \neg\neg\psi(A)
\end{aligned}$$

Combining Kuroda's and Gentzen-Gödel's translations

- How does it work ?

RHS	LHS	Kuroda
$\varphi(P) = \neg\neg P$	$\chi(P) = P$	$\psi(P) = P$
$\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B)$	$\chi(A \wedge B) = \chi(A) \wedge \chi(B)$	$\psi(A \wedge B) = \psi(A) \wedge \psi(B)$
$\varphi(A \vee B) = \neg\neg\psi(A) \vee \psi(B)$	$\chi(A \vee B) = \chi(A) \vee \chi(B)$	$\psi(A \vee B) = \psi(A) \vee \psi(B)$
$\varphi(A \Rightarrow B) = \chi(A) \Rightarrow \varphi(B)$	$\chi(A \Rightarrow B) = \psi(A) \Rightarrow \chi(B)$	$\psi(A \Rightarrow B) = \chi(A) \Rightarrow \psi(B)$
$\varphi(\exists xA) = \neg\neg\exists x\psi(A)$	$\chi(\exists xA) = \exists x\chi(A)$	$\psi(\exists xA) = \exists x\psi(A)$
$\varphi(\forall xA) = \forall x\varphi(A)$	$\chi(\forall xA) = \forall x\chi(A)$	$\psi(\forall xA) = \forall x\varphi(A)$

- How to prove that ? Refine focusing into phases.

Example of translation

$\chi((A \vee B) \Rightarrow C)$ is $(A \vee B) \Rightarrow C$

$\varphi((A \vee B) \Rightarrow C)$ is $(A \vee B) \Rightarrow \neg\neg C$

$$\frac{}{\Gamma, A \vdash \ . \ ; A, \Delta}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \ . \ ; \Delta}{\Gamma, A \wedge B \vdash \ . \ ; \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A \ ; \Delta \qquad \Gamma \vdash B \ ; \Delta}{\Gamma \vdash A \wedge B \ ; \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \ . \ ; \Delta \qquad \Gamma, B \vdash \ . \ ; \Delta}{\Gamma, A \vee B \vdash \ . \ ; \Delta}\ \vee_L \qquad \frac{\Gamma \vdash \ . \ ; A, B, \Delta}{\Gamma \vdash \ . \ ; A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A \ ; \Delta \qquad \Gamma, B \vdash \ . \ ; \Delta}{\Gamma, A \Rightarrow B \vdash \ . \ ; \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B \ ; \Delta}{\Gamma \vdash A \Rightarrow B \ ; \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma, A[c/x] \vdash \ . \ ; \Delta}{\Gamma, \exists x A \vdash \ . \ ; \Delta}\ \exists_L \qquad \frac{\Gamma \vdash \ . \ ; A[t/x], \Delta}{\Gamma \vdash \ . \ ; \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \ . \ ; \Delta}{\Gamma, \forall x A \vdash \ . \ ; \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x] \ ; \Delta}{\Gamma \vdash \forall x A \ ; \Delta}\ \forall_R$$

$$\frac{\Gamma \vdash A \ ; \Delta}{\Gamma \vdash \ . \ ; A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash \ . \ ; A, \Delta}{\Gamma \vdash A \ ; \Delta}\ \text{release}$$

Results

Theorem [Gilbert]

if $\Gamma_0, \neg\Gamma_1 \vdash A; \Delta$ in $LK_{\uparrow\downarrow}$ then $\chi(\Gamma_0), \neg\psi(\Gamma_1), \neg\psi(\Delta) \vdash \varphi(A)$ in LJ.

Theorem [Gilbert]

$A \mapsto \varphi(A)$ is minimal among the $\neg\neg$-translations.

- 58% of Zenon's modulo proofs are secretly constructive
- polarizing the translation of rewrite rules in Deduction modulo:
  - problem with cut elimination: a rule is usable in the lhs and rhs
  - back to a non-polarized one
  - further work: use polarized Deduction modulo
- further work: polarize Krivine's translation

What you hopefully should remember:

- Focusing is a perfect tool to remove double-negations;
- antinegation $\lrcorner$.