From normalization to cut elimination

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Deduction modulo [Dowek, Hardin & Kirchner]

Original idea: *combine automated theorem proving with rewriting*

Generalized to: *combine any first-order deduction process with rewriting*

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Example: Classical Sequent Calculus Modulo

- first-order logic: function and predicate symbols, logical connectors $\wedge, \vee, \Rightarrow$, quantifiers $\forall, \exists$ and constants $\top, \bot$

$$\text{LK} \quad + \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta}\ \textsc{Conv-R} \quad + \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, B \vdash \Delta}\ \textsc{Conv-L}$$

- where Conv rules are applicable whenever $A \equiv B$, the congruence generated by rewriting.

Deduction System I: classical sequent calculus

$$\frac{}{\Gamma, A \vdash A, \Delta} \text{ axiom} \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ cut}$$

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2} \wedge\text{-r} \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\text{-l}$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \Rightarrow\text{-r} \qquad \frac{\Gamma_1, B \vdash \Delta_1 \qquad \Gamma_2 \vdash A, \Delta_2}{\Gamma_1, \Gamma_2, A \Rightarrow B \vdash \Delta_1, \Delta_2} \Rightarrow\text{-l}$$

$$\frac{\Gamma \vdash A[x], \Delta}{\Gamma \vdash \forall x A[x], \Delta} \forall\text{-r, } x \text{ fresh} \qquad \frac{\Gamma, A[t] \vdash \Delta}{\Gamma, \forall x A[x] \vdash \Delta} \forall\text{-l, any } t$$

Deduction System II: intuitionistic natural deduction

$$\frac{}{\Gamma, A \vdash A}\text{axiom}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-i} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-e1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-e2}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow\text{-i} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash A[x]}{\Gamma \vdash \forall x A[x]}\forall\text{-i, } x \text{ free} \qquad \frac{\Gamma \vdash \forall x A[x]}{\Gamma \vdash A[t]}\forall\text{-e, any } t$$

Rewriting relation

- on terms:

$$\begin{array}{rcl} x + 0 & \longrightarrow & x \\ x + S(y) & \longrightarrow & S(x + y) \end{array}$$

- on atomic formulæ:

$$\begin{array}{rcl} Null(0) & \longrightarrow & \top \\ Null(S(x)) & \longrightarrow & \bot \\ A & \longrightarrow & A \Rightarrow A \end{array}$$

(the last one is very bad)

Examples of theories expressed in Deduction Modulo

- arithmetic
- Zermelo's set theory
- a subset of B set theory
- simple type theory (HOL)

What about cut-elimination ?

$$\begin{cases} \vdash \text{even}(0) \\ \text{even}(n) \vdash \text{even}(n+2) \end{cases}$$

$$\text{Cut}\ \frac{\overline{\vdash \text{even}(0)} \qquad \overline{\text{even}(0) \vdash \text{even}(2)}}{\vdash \text{even}(2)}$$

- axiomatic cuts

What about cut-elimination ?

$$\begin{cases} \text{even}(0) & \rightarrow & \top \\ \text{even}(x+2) & \rightarrow & \text{even}(x) \end{cases}$$

$$\dfrac{\overline{\vdash \top}}{\vdash \text{even}(2)}\ \textsc{Conv-r}$$

or even:

$$\dfrac{\overline{\vdash \top}}{\vdash \text{even}(4)}\ \textsc{Conv-r}$$

$$\vdots$$

Cut-elimination implies consistency. . .
and we must pay the prize

Consistency $\Leftarrow$ / $\not\Rightarrow$ Cut elimination $\Leftarrow$ / $\not\Rightarrow$ Normalization

minimal counterexample : $A \to A \Rightarrow B$

convergent counterexample : $\begin{cases} R \in R & \to \forall y. y \simeq R \Rightarrow y \in R \Rightarrow B \\ y \simeq z & \to \forall y. (x \in y \Rightarrow z \in y) \end{cases}$

minimal counterexample : $A \to A \Rightarrow A$

convergent counterexample : $\begin{cases} R \in R & \to \forall y. y \simeq R \Rightarrow y \in R \Rightarrow A \Rightarrow A \\ y \simeq z & \to \forall y. (x \in y \Rightarrow z \in y) \end{cases}$

Normalization: principles

- begin by defining proof-terms and a reduction relation

$$\frac{}{\Gamma, \alpha : A \vdash \alpha : A}\text{axiom}$$

$$\frac{\Gamma \vdash \pi : A \quad \Gamma \vdash \nu : B}{\Gamma \vdash \langle \pi, \nu \rangle : A \wedge B} \wedge\text{-i} \qquad \frac{\Gamma \vdash \pi : A \wedge B}{\Gamma \vdash fst(\pi) : A} \wedge\text{-e1} \qquad \frac{\Gamma \vdash \pi : A \wedge B}{\Gamma \vdash snd(\pi) : B} \wedge\text{-e2}$$

$$\frac{\Gamma, \alpha : A \vdash \pi : B}{\Gamma \vdash \lambda\alpha.\pi : A \Rightarrow B} \Rightarrow\text{-i} \qquad \frac{\Gamma \vdash \pi : A \Rightarrow B \qquad \Gamma \vdash \nu : A}{\Gamma \vdash (\pi\ \nu) : B} \Rightarrow\text{-e}$$

$$\begin{array}{lcr} fst(\langle \pi, \nu \rangle) & \rhd & \pi \\ snd(\langle \pi, \nu \rangle) & \rhd & \nu \\ (\lambda\alpha.\pi\ \nu) & \rhd & (\nu/\alpha)\pi \end{array}$$

- show that every typable proof-term is strongly normalizable

Normalization: principles

- assign to each type A and valuation ϕ a set $[\![A]\!]_\phi$ that is a reducibility candidate. That is, a set $\mathcal{S}$ such that:
  - ⋆ (CR_1) all members of $\mathcal{S}$ are strongly normalizable
  - ⋆ (CR_2) every reduct of $\pi \in \mathcal{S}$ is in $\mathcal{S}$
  - ⋆ (CR_3) if π is neutral[1] and every one-step reduct is in $\mathcal{S}$ then π is in $\mathcal{S}$

[1] an axiom or an elimination / equivalently, a term that, when substituted, does not introduce new redexes

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  - (CR_3) if π is neutral[1] and every one-step reduct is in $\mathcal{S}$ then π is in $\mathcal{S}$

Adequacy

Let $\Gamma \vdash \pi : A$. Let θ be a substitution, ϕ a valuation and σ a substitution for proof variables such that $\sigma(\alpha) \in [\![B]\!]_\phi$ for any $(\alpha : B) \in \Gamma$. Then:

$$\sigma\theta\pi \in [\![A]\!]_\phi$$

- conclusion follows immediately (choose identity for σ and θ)

[1] an axiom or an elimination / equivalently, a term that, when substituted, does not introduce new redexes

Semantics: Heyting algebra

- a universe Ω, operators $\wedge, \vee, \Rightarrow$
- an order $\leq$
- operations on it: lowest upper bound (join: $\wedge$), greatest lower bound (meet: $\vee$ – intersection). A lattice.

$$a \wedge b \leq a \quad a \wedge b \leq b \quad c \leq a \text{ and } c \leq b \text{ implies } c \leq a \wedge b$$
$$a \leq a \vee b \quad b \leq a \vee b \quad a \leq c \text{ and } b \leq c \text{ implies } a \vee b \leq c$$

- like Boolean algebras (classical case), with weaker complement:

$$a \wedge b \leq c \text{ iff } a \leq b \Rightarrow c$$

- example: $\mathbb{R}$ and open sets.

Cut Admissibility: principle

- show that the cut-rule is redundant: we can prove the same statements with of without cuts.
  - this is a consequence of proof normalization
  - more convenient to show (seq. calculus), in any case, simpler argument
  - sometimes we do not have the choice (*cf.* slide 11) !

Cut Admissibility: principle

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  - this is a consequence of proof normalization
  - more convenient to show (seq. calculus), in any case, simpler argument
  - sometimes we do not have the choice (*cf.* slide 11) !
- refinment of soundness/completeness:

Soundness

A provable statement is universally true (for a certain class of models).

Completeness (Gödel)

A universally true (for a certain class of models) statement is provable.

Cut Admissibility: principle

- show that the cut-rule is redundant: we can prove the same statements with of without cuts.
  - ⋆ this is a consequence of proof normalization
  - ⋆ more convenient to show (seq. calculus), in any case, simpler argument
  - ⋆ sometimes we do not have the choice (*cf.* slide 11) !
- refinment of soundness/completeness:

Soundness

A provable statement is universally true (for a certain class of models).

Strong Completeness

A universally true (for a certain class of models) statement is provable **without cut**.

Cut Admissiblity: the Gödel way

- given a context Γ such that $\Gamma \nvdash$ (consistent - say, today, coherent)
- Add all coherent formulæ (whenever $\Gamma, A \nvdash$, add A to Γ - plus Henkin witnesses)
- the limit of this process gives a maximal coherent theory (abstract consistency property)

The syntactical model

Let $[\![A]\!] = 1$ if $A \in \Gamma$ and $[\![A]\!] = 0$ otherwise. This is a model.

- conclude by contradiction:

Completeness theorem

If $\Gamma, \neg\Delta$ does not have a model, then $\Gamma \vdash \Delta$

Cut Admissiblity: the Gödel way

- given a context Γ such that $\Gamma \nvdash^*$ (coherent)
- Add all coherent formulæ (whenever $\Gamma, A \nvdash$, add A to Γ - plus Henkin witnesses)
- the limit of this process gives a maximal coherent theory (abstract consistency property)

The syntactical model

Let $[\![A]\!] = 1$ if $A \in \Gamma$ and $[\![A]\!] = 0$ otherwise. This is a model.

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Completeness theorem

If $\Gamma, \neg\Delta$ does not have a model, then $\Gamma \vdash^* \Delta$

Cut Admissibility: the Gödel way

Extensions:

- Krivine's proof (constructive, classical logic)
- the tableau method (more constructive, cut admissibility)
- Herbelin and Ilik's proofs (even more constructive: proved in Coq)
- intuitionistic logic: Kripke structures (constructive versions by Freidman, Veldman)
- Normalization by Evaluation ...

Cut Admissibility: the Okada way

Lindenbaum algebra:

- $\lfloor A \rfloor = \{B \mid A \vdash B \text{ and } B \vdash A\}$
- $\Omega = \{\lfloor A \rfloor \mid A \text{ formula}\}$
- $\leq$ is $\vdash$: $\lfloor A \rfloor \leq \lfloor B \rfloor$ iff $A \vdash B$.

Lemma

It is independent of the chosen element of $\lfloor A \rfloor$.

- $\lfloor A \rfloor \wedge \lfloor B \rfloor$ is $\lfloor A \wedge B \rfloor$ (same for other connectives)

Lemma

It is independent of the chosen element of $\lfloor A \rfloor$.

Theorem

$\Omega, \leq, \wedge, \vee, \Rightarrow, \top, \bot, \forall, \exists$ is a Boolean/Heyting Algebra

Cut Admissibility: the Okada way

Lidenbaum algebra

- interpretation of formulæ
  - define the interpretation on the atoms as $[\![A]\!] = \lfloor A \rfloor$
  - extend it by induction

Fundamental Lemma

For *any* formula A, $[\![A]\!] = \lfloor A \rfloor$

- what do we have ?

Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash B$.

  - this is the definition of $\leq$ in the Lindenbaum algebra.

- defining $\lfloor A \rfloor = \{B \mid A \vdash^* B \text{ and } B \vdash^* A\}$ does not work (transitivity of $\leq$ fails)

Cut Admissibility: the Okada way

Base elements of the Lindenbaum algebra

$\lfloor A \rfloor = \{B \mid A \vdash B \text{ and } B \vdash A\}$

Cut Admissibility: the Okada way

Base elements of the context algebra

$\lfloor A \rfloor = \{\Gamma \mid \Gamma \vdash A\}$

- $\leq$ is $\subseteq$ and g.l.b. ($\wedge$) and l.u.b. ($\vee$) will be "intersection" and "union"
- implies changes in the approach:

The Algebra Ω

$$\Omega = \left\{ \bigcap_{C \in \mathcal{C}} \lfloor C \rfloor \mid \text{ for } \mathcal{C} \text{ set of formulæ} \right\}$$

Ω is composed of arbitrary intersections of base elements

The context algebra for completeness

- lattice operators:

$$
\begin{aligned}
\top &= \lfloor \top \rfloor = \{\Gamma \mid \Gamma \text{ valid context}\} \\
\bot &= \lfloor \bot \rfloor = \{\Gamma \mid \Gamma \vdash \bot\} \\
a \wedge b &= a \cap b \\
a \vee b &= \bigcap\{d \in \Omega \mid a \cup b \subseteq d\} = \bigcap\{\lfloor D \rfloor \mid a \cup b \subseteq \lfloor D \rfloor\} \\
\forall\, \mathcal{S} &= \bigcap \mathcal{S} = \bigcap_{s \in \mathcal{S}} s \\
\exists\, \mathcal{S} &= \bigcap\{d \in \Omega \mid (\bigcup \mathcal{S}) \subseteq d\} = \bigcap\{\lfloor D \rfloor \mid (\bigcup \mathcal{S}) \subseteq \lfloor D \rfloor\}
\end{aligned}
$$

Lemma: Ω is a lattice

$\wedge, \forall, \vee, \exists$ represent the binary greatest lower bound, greatest lower bound, binary least upper bound and least upper bound respectively. $\top$ and $\bot$ are the greatest and lowest elements, respectively.

- it is also a Boolean/Heyting Algebra.

Cut Admissibility: the Okada way

- set the interpretation of the atoms to be: $[\![A]\!] = \lfloor A \rfloor$

Fundamental Lemma

For *any* formula A, $[\![A]\!] = \lfloor A \rfloor$.

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Fundamental Lemma

For *any* formula A, $[\![A]\!] = \lfloor A \rfloor$.

- what do we have ?

Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash B$.

- ★ (trivial) $A \in \lfloor A \rfloor$
- ★ $\lfloor A \rfloor = [\![A]\!]$ (fundamental lemma)
- ★ $A \in [\![B]\!] = \lfloor B \rfloor$ (fundamental lemma)
- ★ means $A \vdash B$

Cut Admissibility: the Okada way

- Ω is arbitrary intersections of base elements.

Base elements

$\lfloor A \rfloor = \{\Gamma \mid \Gamma \vdash A\}$

- $\leq$ is $\subseteq$. Gives a lattice.
- it is also a Boolean/Heyting algebra (phase space).
- set the interpretation of the atoms to be: $[\![A]\!] = \lfloor A \rfloor$

Fundamental Lemma

For *any* formula A, $[\![A]\!] = \lfloor A \rfloor$.

- what do we have ?

Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash B$.

Proof: $A \in \lfloor A \rfloor = [\![A]\!] \subseteq [\![B]\!] = \lfloor B \rfloor$.

Cut Admissibility: the Okada way

- Ω is arbitrary intersections of base elements.

Base elements

$\lfloor A \rfloor = \{\Gamma \mid \Gamma \vdash^* A\}$

- $\leq$ is $\subseteq$. Gives a lattice.
- it is also a Boolean/Heyting algebra (original work: phase space).
- set the interpretation of the atoms to be: $[\![A]\!] = \lfloor A \rfloor$

Fundamental Lemma

For *any* formula A, $A \in [\![A]\!] \subseteq \lfloor A \rfloor$

- what do we have ?

Strong Completeness

if $[\![A]\!] \leq [\![B]\!]$ in all models, then $A \vdash^* B$.

Proof: $A \in [\![A]\!] \subseteq [\![B]\!] \subseteq \lfloor B \rfloor$.

In Deduction Modulo ...

Congruence

The congruence generated by the rewriting relation is a condition for strict equality.

- for models, we impose $A \equiv B$ implies $[\![A]\!] = [\![B]\!]$
- same for reducibility candidates (although $[\![\top]\!] \neq [\![\top \wedge \top]\!]$)

end of the introduction

Reducibility candidates for cut admissibility

- considered deduction system: natural deduction
- principle: drop the proof-terms (we do not care about normalization) and replace them with their conclusion (a sequent).
- redefine what "cut-free" means:

Cut-free

A proof:
that ends with an axiom ; that ends with an introduction which premises are proved cut-free ; that ends with an elimination which principal premiss is proved neutrally and cut-free, and other premises are proved cut-free is cut-free

- condition for a set of sequents $\mathcal{S}$ to be a reducibility candidate:
  - (CR_1) containing only cut-free provable sequents
  - no CR_2 (stability by reduction)
  - (CR_3) contain all the sequents provable with a neutral cut-free proof

Building enough candidates

Operators - the S-algebra

Let a, b bet sets of sequents:

- $\top$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or such that $C \equiv \top$
- $a \wedge b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or such that $C \equiv A \wedge B$ and $\Gamma \vdash A \in a$ and $\Gamma \vdash B \in b$
- ...

- to each formula A and valuation ϕ, we shall associate a candidate $[\![A]\!]_\phi$:
  - A atomic: $[\![A]\!]_\phi$ chosen arbitrarily (depending on ϕ, however)
  - A compound: $[\![B \wedge C]\!]_\phi = [\![B]\!]_\phi \wedge [\![C]\!]_\phi, \cdots$
- in Deduction modulo, not sufficient:

if $A \equiv B$ then $[\![A]\!] = [\![B]\!]$

- it looks like a model interpretation, let it be really like this.

Chosing a candidate for atomic formulæ: superconsistency (SC), a generic criterion

Dowek & Werner: *Proof normalization modulo*
Dowek: *Truth values algebras and proof normalization*

Consistency

A theory $\mathcal{T}$ is consistent if it can be interpreted in **one** model not reduced to $\bot$

Super-consistency

A theory $\mathcal{T}$ is super-consistent if it can be interpreted in **all** models

What is the notion of model ?

Pre-Heyting Algebras

... are Heyting algebras generalized to *pre-ordered sets*

Pre-Heyting algebras take into account two distinct notion of equivalence:

Computational equivalence : strong, corresponds to equality in the model

Logical equivalence : loose corresponds to $\geqslant \cap \leqslant$

We also can look at Pre-Heyting algebra as an algebra with operators (drop entirely the pre-order)

Superconsistency (SC): characterizing analytical theories

Dowek's remark

The set of reducibility candidates for NJ modulo is a pre-Heyting Algebra. And the normalization constructions do not depend on the specificities of the reducibility algebra: we can abstract and generalize.

Superconsistency (SC): characterizing analytical theories

Dowek's remark

The set of reducibility candidates for NJ modulo is a pre-Heyting Algebra. And the normalization constructions do not depend on the specificities of the reducibility algebra: we can abstract and generalize.

Consistency The theory can be interpreted in a non-trivial model

Superconsistency The theory can be interpreted in any model

Any superconsistent theory can then be interpreted in the pre-Heyting algebra of reducibility candidates. Using generic adequacy:

Conclusion

Any superconsistent theory is strongly normalizable (for NJ)

Examples of theories proved to be superconsistent

- arithmetic
- simple type theory (HOL)
- confluent, terminating and quantifier free rewrite systems
- confluent, terminating and positive rewrite systems
- positive rewrite system such that each atomic formula has at most one one-step reduct

Back to the S-algebra and adequacy

Operators - the S-algebra

Let a, b bet sets of sequents:

- $\top$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or such that $C \equiv \top$
- $a \wedge b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut-free proof or such that $C \equiv A \wedge B$ and $\Gamma \vdash A \in a$ and $\Gamma \vdash B \in b$
- ...

- it is a pre-Heyting algebra, but not a Heyting algebra: $[\![\top \wedge \top]\!]$ contains $\vdash \top \wedge \top$ while $[\![\top]\!]$ does not.
- given a superconsistent theory, we get a model ... but it remains to show adequacy in this setting.

A hidden Heyting algebra

- we assume a sequent reducibility candidates model $\mathcal{M}$.

Context extraction

$\lfloor A \rfloor$ is the set of contexts Γ such that for any substitution σ and valuation ϕ, and any context Δ such that $\Delta \vdash \sigma A_i \in [\![A_i]\!]_\phi$ for any $A_i \in \Gamma$, then $\Delta \vdash \sigma A \in [\![A]\!]_\phi$

Reminder: (old proof-term) adequacy

[...] Let σ be a substitution, ϕ a valuation and δ a substitution for proof variables such that $\delta(\alpha) \in [\![A_i]\!]_\phi$ for any $(\alpha : A_i) \in \Gamma$ [...]

- we define the following:
  - ⋆ Ω is the least set containing the extractions and closed by arbitrary intersection
  - ⋆ this forms a lattice
  - ⋆ we can extend it to a Heyting algebra

Fundamental Lemma and Adequacy

Fundamental lemma

- $\lfloor A \wedge B \rfloor = \lfloor A \rfloor \wedge \lfloor B \rfloor$
- $\lfloor A \Rightarrow B \rfloor = \lfloor A \rfloor \Rightarrow \lfloor B \rfloor$
- ...

Remarks:

- not "self-evident" (semantic $\wedge$ is the intersection)
- other fundamental lemmata: $A \wedge B \in [\![A]\!] \wedge [\![B]\!] \subseteq \lfloor A \rfloor \wedge \lfloor B \rfloor \subseteq \lfloor A \wedge B \rfloor$ (impossible to do otherwise)
- all the case mimic the cases of adequacy lemma: but (of course) no induction hypothesis application.

Regaining cut admissibility

- build a second level of Heyting-valued model $\mathcal{D}$, where $[\![A]\!]^{\mathcal{D}} = \lfloor A \rfloor$ and terms are interpreted by themselves (equivalence classes modulo $\equiv$).

Cut Admissibility

if $A \vdash B$ is provable, it has a cut-free proof.

- interpret it in $\mathcal{D}$: $\lfloor A \rfloor \subseteq \lfloor B \rfloor$ (soundness)
- but $A \in \lfloor A \rfloor$
- so $A \in \lfloor B \rfloor$
- and $A \vdash B \in [\![B]\!]^{\mathcal{S}}$
- then $A \vdash B$ has a cut-free proof

Remark:

- compared to adequacy proof, induction handled by soundness and inductive cases by Fundamental lemma (hidden here)

To Summarize

- construct a pre-Heyting algebra made of sets of sequents
- interpret propositions inside this algebra (thanks to SC)
- extract a Heyting algebra and a model interpretation / show adequacy
- soundness + completeness: cut-elimination

Application to the HOL case

- HOL as a first-order theory: Deduction modulo
- we build the second level of model $\mathcal{D}$ as usual. In particular, terms are interpreted by (equivalence classes of) terms.
- all the other cut admissibility proofs introduce a weird device, V-complexes, due to the intensionality problem:

Example of intensionality

$P(\top \wedge \top) \not\Leftrightarrow P(\top)$ although $\top \wedge \top \Leftrightarrow \top$

- ⋆ $\top$ must be interpreted by something else that the semantic $\top$
- ⋆ need for a new notion of model, with two layers: the interpretation one (V-complexes, pairs $\langle t, d \rangle$) and the denotation one (logical meaning).

Application to the HOL case

We do not need to change the notion of model, for two reasons (both necessary):

- there is a propositional $\top = \varepsilon(\dot{\top})$ and a term-level $\top$: $\dot{\top}$. They are different.
- the sequent algebra is richer than a Boolean/Heyting algebra: $[\![\top \wedge \top]\!] \neq [\![\top]\!]$. They can be distinguished.

Simplification and explanation of old arguments (Takahashi, Prawitz, Andrews).

Classical sequent calculus

- completely different notion of cut !
- aim: define and use SC for (eventually) sequent-calculus proof-terms
- framework: one-sided sequent calculus, negation as an operator (not a connective)

$$\frac{}{\vdash A, A^{\perp}}\ (\text{Axiom}) \qquad \frac{\vdash A, \Delta_1 \qquad \vdash A^{\perp}, \Delta_2}{\vdash \Delta_1, \Delta_2}\ (\text{Cut}) \qquad \frac{\vdash A, \Delta \qquad A \equiv B}{\vdash B, \Delta}\ (\text{Cor}$$

$$\frac{\vdash A, A, \Delta}{\vdash A, \Delta}\ (\text{Contr}) \qquad \frac{\vdash \Delta}{\vdash A, \Delta}\ (\text{Weak}) \qquad \frac{}{\vdash \top}\ (\top) \qquad (\text{no rule for } \bot)$$

$$\frac{\vdash A, \Delta_1 \qquad \vdash B, \Delta_2}{\vdash A \wedge B, \Delta_1, \Delta_2}\ (\wedge) \qquad \frac{\vdash A, B, \Delta}{\vdash A \vee B, \Delta}\ (\vee)$$

$$\frac{\vdash A[t/x], \Delta}{\vdash \exists x.A, \Delta}\ (\exists) \qquad \frac{\vdash A, \Delta \qquad x \text{ fresh in } \Delta}{\vdash \forall x.A, \Delta}\ (\forall)$$

A road map/recipe

Suppose you have an unspecified superconsistent theory

Step 1 Construct a set of reducibility candidates

Step 2 Prove that it is a pre-Boolean algebra

you get an interpretation of sequents in the algebra for free thanks to superconsistency (adapted to Boolean algebra)

Step 3 Prove adequacy: provable sequents are in their interpretations

you get cut-elimination as a direct corollary

Inheritance from Linear Logic [Okada, Brunel]

- identifying a site in sequents: pointed sequents

$$\vdash \Delta, A^\circ$$

- interaction: a partial function $\star$

$$\vdash \Delta_1, A^\circ \;\star\; \vdash \Delta_2, B^\circ = \;\vdash \Delta_1, \Delta_2 \qquad \text{if } A \equiv B^\perp$$

$$\vdash \Delta_1, A^\circ \;\star\; X = \{ \vdash \Delta_1, \Delta_2 \mid \;\vdash \Delta_2, B^\circ \in X \text{ and } A \equiv B^\perp \}$$

- define an object having good properties: $⫫$

  the set of cut-free provable sequents in $LK_\equiv$

- define an orthogonality operation on sets of sequents:

$$X^\perp = \{ \vdash \Delta, A^\circ \mid \;\vdash \Delta, A^\circ \;\star\; X \subseteq ⫫ \}$$

  - usual properties of an orthogonality operation:

$$X \subseteq X^{\perp\perp} \qquad X \subseteq Y \Rightarrow Y^\perp \subseteq X^\perp \qquad X^{\perp\perp\perp} = X^\perp$$

Step 1: construct the set of reducibility candidates

- the domain of interpretation D: set of sequents

$$Ax^\circ \subseteq X \subseteq ⫫^\circ$$

which are behaviours: $X^{\perp\perp} = X$

- reducibility candidates analogy:

CR1 $X \subseteq ⫫$ (cut-free provable sequents / SN proofterms)
CR2 none (no reduction)
CR3 $Ax^\circ \subseteq X$ (neutral proofterms)

- core operation + orthogonality:

$$\begin{aligned} X.Y &= \{ \vdash \Delta_A, \Delta_B, (A \wedge B)^\circ \mid (\vdash \Delta_A, A^\circ) \in X \\ &\qquad \text{and } (\vdash \Delta_B, B^\circ) \in Y \} \\ X \wedge Y &= \{X.Y \cup Ax^\circ\}^{\perp\perp} \end{aligned}$$

Step 2: prove that it is a pre-Boolean algebra

D forms a pre-Boolean algebra:

- cheat on $\leq$: take the trivial pre-order
  - ⋆ we can even drop it in the definition (see slide 35)
- stability of D under $(.)^{\perp}$, $\wedge$
- stability of elements of D under $\equiv$

Step 3: prove adequacy

Super-consistency:

- give us an interpretation such that $A \equiv B$ implies $A^* = B^*$

Adequacy:

- takes a proof of $\vdash A_1, ..., A_n$
- assumes $\vdash \Delta_i, (A_i^\perp)^\circ \in A_i^{*\perp}$
- ensures $\vdash \Delta_1, ..., \Delta_n \in \perp\!\!\!\perp$

Features of the theorem:

- conversion rule: processed by the SC condition

Directly implies cut-elimination:

- because $Ax^\circ \subseteq A_i^{*\perp}$ (untyped candidates), we have $\vdash A, (A^\perp)^\circ \in A_i^{*\perp}$
- because of the definition of $\perp\!\!\!\perp$ (cut-free provable sequents)

We can also extract a Boolean algebra.

Towards NbE (work in progress ...)

- we can do a similar work with proof-terms

Context extraction

$\lfloor A \rfloor$ is composed of the Γ such that there exists a proof-term $\Gamma \vdash \pi : A$ (variant: in normal form) and for any valuation ϕ, substitution θ, and assignment σ assigning to any $\alpha : A \in \Gamma$ a value $\sigma\alpha \in [\![A]\!]_\phi$, we have:

$$\sigma\theta\pi \in [\![A]\!]_\phi$$

- similar reasoning leads to a proof in normal form
- ... but we lost π in the way (soundness made π become a *justification* at the Meta-Level - completeness cannot make it go down).
- the NF we get is $\downarrow \pi$. Visible, but not provable.

Conclusion

- carry π all the way ?
- Heyting towards Kripke ?
  - NbE works are in Kripke style
  - Herbelin and Iliḱ's work
- SC for Heyting implies SC for Boole: does the converse stand ?
- what about normalization in $LK_{\equiv}$ by SC ?