Introduction to Computer Systems

15-213/18-243, spring 2009

4th Lecture, Jan. 22nd

Instructors:

Gregory Kesden and Markus Püschel

Last Time: Integers

- **Representation: unsigned and signed**
- **Conversion, casting**
  - Bit representation maintained but reinterpreted
- **Expanding, truncating**
  - Truncating = mod
- **Addition, negation, multiplication, shifting**
  - Operations are mod 2^w
- **"Ring" properties hold**
  - Associative, commutative, distributive, additive 0 and inverse
- **Ordering properties do not hold**
  - $u > 0$ does not mean $u + v > v$
  - $u, v > 0$ does not mean $u \cdot v > 0$

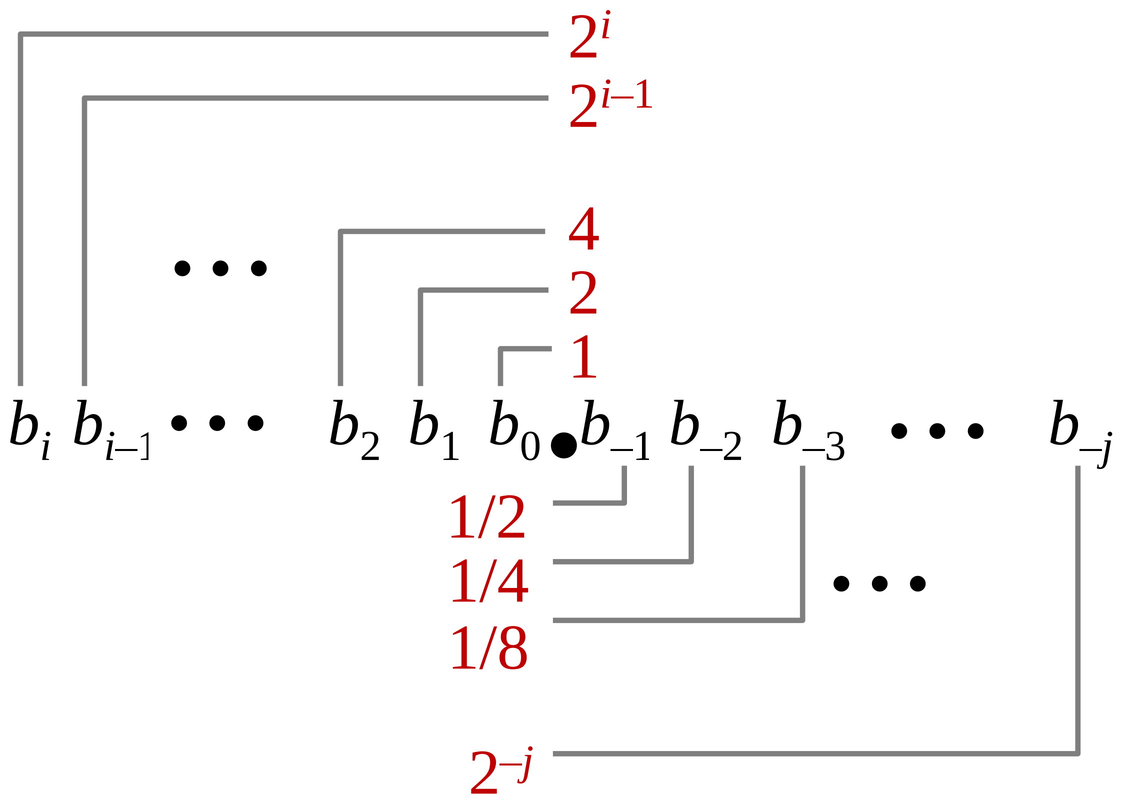
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101?

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \cdot 2^k$$

Fractional Binary Numbers: Examples

Value	Representation
5-3/4	$\texttt{101.11}_2$
2-7/8	$\texttt{10.111}_2$
63/64	$\texttt{0.111111}_2$

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $\texttt{0.111111}\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

- **Value** **Representation**

Value	Representation
1/3	`0.0101010101[01]`…$_2$
1/5	`0.001100110011[0011]`…$_2$
1/10	`0.0001100110011[0011]`…$_2$

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IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

  $$(-1)^s \, M \; 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand *M*** normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two

- **Encoding**
  - MSB **`s`** is sign bit *s*
  - **`exp`** field encodes *E* (but is not equal to E)
  - **`frac`** field encodes *M* (but is not equal to M)

`s`	`exp`	`frac`

Precisions

- **Single precision: 32 bits**

`s`	`exp`	`frac`
1	8	23

- **Double precision: 64 bits**

`s`	`exp`	`frac`
1	11	52

- **Extended precision: 80 bits (Intel only)**

`s`	`exp`	`frac`
1	15	63 or 64

Normalized Values

- **Condition: `exp` $\neq$ `000…0` and `exp` $\neq$ `111…1`**

- **Exponent coded as *biased* value: *E* = *Exp* – *Bias***
  - *Exp*: unsigned value **`exp`**
  - *Bias* = $2^{e-1} - 1$, where e is number of exponent bits
    - Single precision: 127 (*Exp*: 1…254, *E*: -126…127)
    - Double precision: 1023 (*Exp*: 1…2046, *E*: -1022…1023)

- **Significand coded with implied leading 1: *M* = `1.xxx…x`$_2$**
  - **`xxx…x`**: bits of **`frac`**
  - Minimum when **`000…0`** ($M = 1.0$)
  - Maximum when **`111…1`** ($M = 2.0 - \varepsilon$)
  - Get extra leading bit for "free"

Normalized Encoding Example

- **Value:** `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = `1.`$\underline{\texttt{1101101101101}}_2$

  `frac`= $\underline{\texttt{1101101101101}}\texttt{0000000000}_2$

- **Exponent**

  E = 13

  $Bias$ = 127

  Exp = 140 = $\texttt{10001100}_2$

- **Result:**

`0`	`10001100`	`11011011011010000000000`
`s`	**`exp`**	**`frac`**

Denormalized Values

- **Condition: `exp` = 000…0**

- **Exponent value: *E* = –*Bias* + 1 (instead of E = 0 – Bias)**
- **Significand coded with implied leading 0: *M* = 0 . $\mathtt{xxx…x}_2$**
    - **`xxx…x`**: bits of **`frac`**
- **Cases**
    - **`exp` = 000…0, `frac` = 000…0**
        - Represents value 0
        - Note distinct values: +0 and –0 (why?)
    - **`exp` = 000…0, `frac` ≠ 000…0**
        - Numbers very close to 0.0
        - Lose precision as get smaller
        - Equispaced

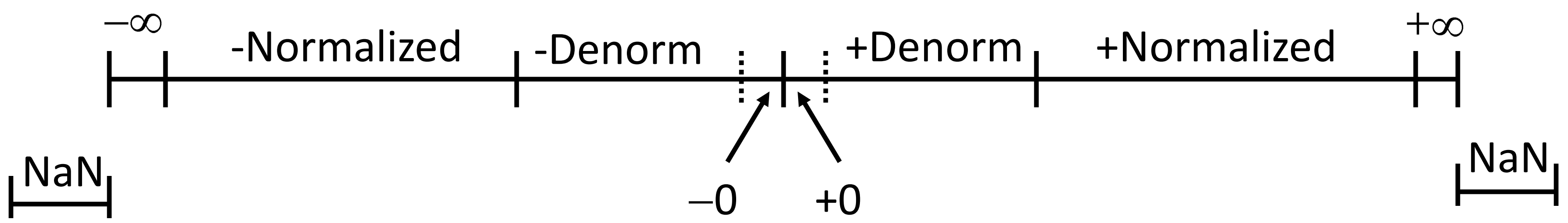
Special Values

- **Condition: `exp` = `111…1`**

- **Case: `exp` = `111…1`, `frac` = `000…0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., $1.0/0.0 = -1.0/-0.0 = +\infty$, $1.0/-0.0 = -\infty$

- **Case: `exp` = `111…1`, `frac` $\neq$ `000…0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), $\infty - \infty$, $\infty * 0$

Visualization: Floating Point Encodings



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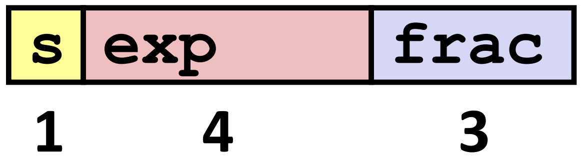
Tiny Floating Point Example



- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the **`frac`**

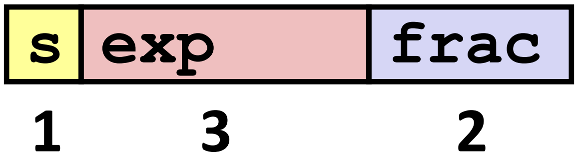
- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

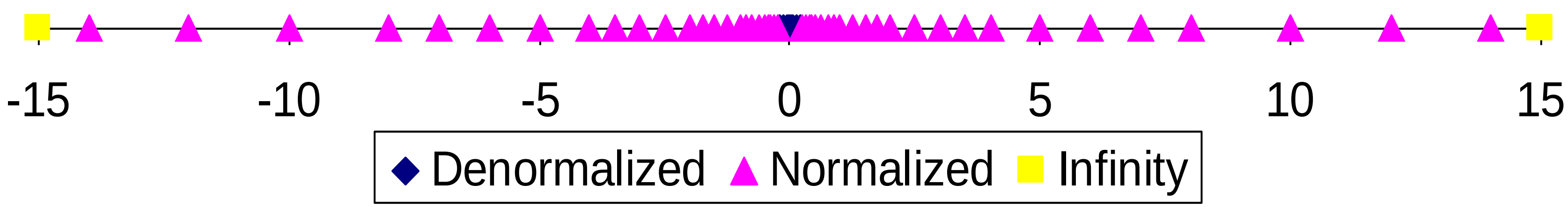
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

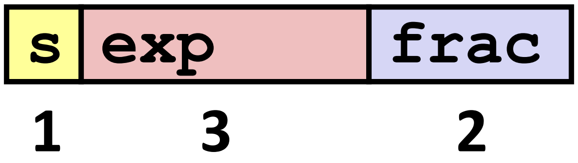


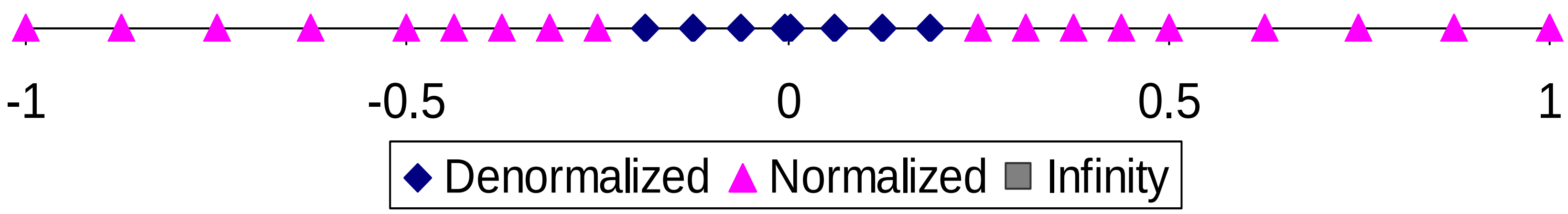
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Interesting Numbers {single,double}

Description	exp	frac	Numeric Value
■ Zero	00…00	00…00	0.0
■ Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
■ Single ≈ 1.4×10^{-45}			
■ Double ≈ 4.9×10^{-324}			
■ Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
■ Single ≈ 1.18×10^{-38}			
■ Double ≈ 2.2×10^{-308}			
■ Smallest Pos. Normalized		00…01	00…00 $1.0 \times 2^{-\{126,1022\}}$
■ Just larger than largest denormalized			
■ One	01…11	00…00	1.0
■ Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
■ Single ≈ 3.4×10^{38}			
■ Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0

- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider -0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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- **Rounding, addition, multiplication**
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Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$
- $x \times_f y = \text{Round}(x \times y)$
- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
Towards zero	$1	$1	$1	$2	–$1
Round down (-∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

- **What are the advantages of the modes?**

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

$(-1)^{s1}\ M1\ \ 2^{E1}\ \ \mathbf{x}\ \ (-1)^{s2}\ M2\ \ 2^{E2}$

- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign *s*: $s1$ ^ $s2$
  - Significand *M*: $M1 * M2$
  - Exponent *E*: $E1 + E2$

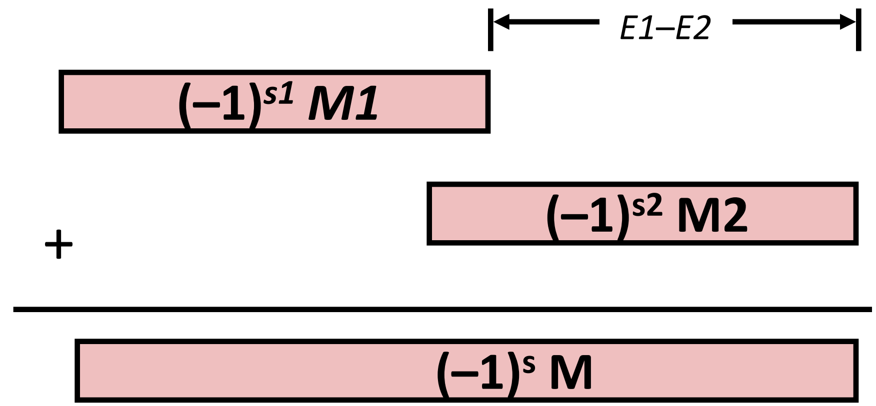
- **Fixing**
  - If $M \geq 2$, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit `frac` precision

- **Implementation**
  - Biggest chore is multiplying significands

Floating Point Addition

$(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$

Assume $E1 > E2$



- **Exact Result: $(-1)^s\ M\ \ 2^E$**
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: $E1$
- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? ***Yes***
    - But may generate infinity or NaN
  - Commutative? ***Yes***
  - Associative? ***No***
    - Overflow and inexactness of rounding
  - 0 is additive identity? ***Yes***
  - Every element has additive inverse ***Almost***
    - Except for infinities & NaNs
- **Monotonicity**
  - $a \geq b \Rightarrow a+c \geq b+c$? ***Almost***
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication? ***Yes***
    - But may generate infinity or NaN
  - Multiplication Commutative? ***Yes***
  - Multiplication is Associative? ***No***
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity? ***Yes***
  - Multiplication distributes over addition? ***No***
    - Possibility of overflow, inexactness of rounding

- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? ***Almost***
    - Except for infinities & NaNs

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Floating Point in C

- **C Guarantees Two Levels**

  `float` single precision

  `double` double precision

- **Conversions/Casting**
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`Double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as int has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;

float f = …;

double d = …;
```

Assume neither `d` nor `f` is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

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Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form $M \times 2^{E}$**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

More Slides

Creating Floating Point Number

- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

7	6 … 3	2 … 0
`s`	`exp`	`frac`

- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format
  - Example Numbers

128	**10000000**
15	**00001101**
33	**00010001**
35	**00010011**
138	**10001010**
63	**00111111**

Normalize

7	6 … 3	2 … 0
`s`	`exp`	`frac`

- **Requirement**
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
128	`10000000`	`1.0000000`	`7`
15	`00001101`	`1.1010000`	`3`
17	`00010001`	`1.0001000`	`5`
19	`00010011`	`1.0011000`	`5`
138	`10001010`	`1.0001010`	`7`
63	`00111111`	`1.1111100`	`5`

Rounding

1.BBGRXXX

Guard bit: LSB of result

ound bit: 1st bit removed

Sticky bit: OR of remaining b

- **Round up conditions**
  - Round = 1, Sticky = 1 ➔ > 0.5
  - Guard = 1, Round = 1, Sticky = 0 ➔ Round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	`1.000`	7		`128`
15	`1.101`	3		`15`
17	`1.000`	4		`16`
19	`1.010`	4		`20`
138	`1.001`	7		`134`
63	`10.000`	5	`1.000/6`	`64`