

# Double Negation Translations as Morphisms

Olivier Hermant

CRI, MINES ParisTech

December 12, 2014

Deducteam Seminar, INRIA

# Double-Negation Translations

Double-Negation translations:

- ▶ a shallow way to encode classical logic into intuitionistic
- ▶ Zenon's backend for Dedukti
- ▶ existing translations: Kolmogorov's (1925), Gentzen-Gödel's (1933), Kuroda's (1951), Krivine's (1990), ...

Minimizing the translations:

- ▶ turns more formulæ into themselves;
- ▶ shifts a classical proof into an intuitionistic proof of the *same* formula.

# Morphisms

- ▶ A morphism preserves the operations between two structures:

$$\text{Group morphism: } \begin{cases} (\mathbb{Z}, +, 0) \mapsto (\mathbb{R}^*, *, 1) \\ h(0) \rightarrow 1 \\ h(a + b) \rightarrow h(a) * h(b) \end{cases}$$

- ▶ a translation that is a morphism:

$$\begin{aligned} h(P) &= P \\ h(A \wedge B) &= h(A) \wedge h(B) \\ h(A \vee B) &= h(A) \vee h(B) \\ h(A \Rightarrow B) &= h(A) \Rightarrow h(B) \\ h(\forall x A) &= \forall x h(A) \\ h(\exists x A) &= \exists x h(A) \end{aligned}$$

(of course this is the **identity**)

# Morphisms

- ▶ A morphism preserves the operations between two structures:

$$\text{Group morphism: } \begin{cases} (\mathbb{Z}, +, 0) \mapsto (\mathbb{R}^*, *, 1) \\ h(0) \rightarrow 1 \\ h(a + b) \rightarrow h(a) * h(b) \end{cases}$$

- ▶ a **more interesting** translation that is a morphism:

$$\begin{aligned} h(P) &= P \\ h(A \wedge B) &= h(A) \wedge_c h(B) \\ h(A \vee B) &= h(A) \vee_c h(B) \\ h(A \Rightarrow B) &= h(A) \Rightarrow_c h(B) \\ h(\forall x A) &= \forall_c x h(A) \\ h(\exists x A) &= \exists_c x h(A) \end{aligned}$$

**two kinds** of connectives: the **classical** and the **intuitionistic** ones.

# Morphisms

- ▶ A morphism preserves the operations between two structures:

$$\text{Group morphism: } \begin{cases} (\mathbb{Z}, +, 0) & \mapsto & (\mathbb{R}^*, *, 1) \\ h(0) & \rightarrow & 1 \\ h(a + b) & \rightarrow & h(a) * h(b) \end{cases}$$

- ▶ a **more interesting** translation that is a morphism:

$$\begin{aligned} h(P) &= P \\ h(A \wedge B) &= h(A) \wedge_c h(B) \\ h(A \vee B) &= h(A) \vee_c h(B) \\ h(A \Rightarrow B) &= h(A) \Rightarrow_c h(B) \\ h(\forall x A) &= \forall_c x h(A) \\ h(\exists x A) &= \exists_c x h(A) \end{aligned}$$

**two kinds** of connectives: the **classical** and the **intuitionistic** ones.

- ▶ Design a **unified** logic, where we can reason both classically and intuitionistically:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee_i B}$$

$$\frac{\text{strange premises}}{\Gamma \vdash A \vee_c B}$$

# Translations that are Morphisms

- ▶ None of the previous translations is a morphism.
- ▶ Dowek has shown one, it is very verbose.
- ▶ We make it lighter.

Plan:

- 1 Classical and Intuitionistic Logic
- 2 Sequent Calculus
- 3 Double Negation Translations
- 4 Morphisms

# Classical vs. Intuitionistic

- ▶ The principle of excluded-middle. Should

$$A \vee \neg A$$

be provable ? Yes or no ?

- ▶ **Yes**. This is what is called **classical** logic.
- ▶ Wait a minute !

## The Drinker's Principle

In a bar, there is somebody such that, if he drinks, then everybody drinks.

## Two Irrationals

There exists  $i_1, i_2 \in \mathbb{R} \setminus \mathbb{Q}$  such that  $i_1^{i_2} \in \mathbb{Q}$ .

## IRL: A Manicchean World

You are with us, or against us (G. W. Bush)

Rashomon (A. Kurosawa)

# Classical vs. Intuitionistic

- ▶ The principle of excluded-middle. Should

$$A \vee \neg A$$

be provable ? Yes or no ?

- ▶ **No**. This is the **constructivist** school (Brouwer, Heyting, Kolmogorov).
- ▶ Intuitionistic logic is one of those branches. It features the BHK interpretation of proofs:

## Witness Property

A proof of  $\exists xA$  (in the empty context) gives a **witness**  $t$  for the property  $A$ .

## Disjunction Property

A proof of  $A \vee B$  (in the empty context) reduces eventually **either** to a proof of  $A$ , **or** to a proof of  $B$ .



# The Classical Sequent Calculus (LK)

$$\frac{}{\Gamma, A \vdash A, \Delta} \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow_L$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \Rightarrow_R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg_L$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \exists_L$$

$$\frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \forall_L$$

$$\frac{\Gamma \vdash A[c/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \forall_R$$

# The Intuitionistic Sequent Calculus (LJ)

$$\frac{}{\Gamma, A \vdash A} \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee_{R1} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee_{R2}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow_L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_R$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash \Delta} \neg_L$$

$$\frac{\Gamma, A \vdash}{\Gamma \vdash \neg A} \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \exists_L$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \forall_L$$

$$\frac{\Gamma \vdash A[c/x]}{\Gamma \vdash \forall x A} \forall_R$$

# Note on Frameworks

- ▶ structural rules are not shown (contraction, weakening)
- ▶ left-rules seem **very** similar in both cases
- ▶ so, lhs formulæ can be translated by themselves
- ▶ this accounts for **polarizing** the translations
- ▶ another work [Boudard & H]:
  - ★ does not behave well in presence of cuts
  - ★ appeals to **focusing** techniques

# Examples

- ▶ proofs that behave identically in classical/intuitionistic logic:

$$\frac{\overline{A, B \vdash A} \text{ ax}}{A \vdash B \Rightarrow A} \Rightarrow_R \quad \frac{\overline{A, B \vdash B} \text{ ax}}{\frac{A \wedge B \vdash B}{A \wedge B \vdash B \vee C} \wedge_L} \vee_R$$

- ▶ proof of the excluded-middle:

Classical Logic	Intuitionistic Logic
$\frac{\overline{A \vdash A} \text{ ax}}{\frac{\vdash A, \neg A}{\vdash A \vee \neg A} \neg_R} \vee_R$	

# Examples

- ▶ proofs that behave identically in classical/intuitionistic logic:

$$\frac{\overline{A, B \vdash A} \text{ ax}}{A \vdash B \Rightarrow A} \Rightarrow_R \quad \frac{\overline{A, B \vdash B} \text{ ax}}{\frac{A \wedge B \vdash B}{A \wedge B \vdash B \vee C} \wedge_L} \vee_R$$

- ▶ proof of the excluded-middle:

Classical Logic	Intuitionistic Logic
$\frac{\overline{A \vdash A} \text{ ax}}{\frac{\vdash A, \neg A}{\vdash A \vee \neg A} \neg_R} \vee_R$	$\frac{??}{\vdash A \vee \neg A}$

# The Excluded-Middle in Intuitionistic Logic

- ▶ is not provable. However, its **negation** is inconsistent.

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{ax}{A \vdash A}}{A \vdash A \vee \neg A}}{\neg(A \vee \neg A), A \vdash}}{\neg(A \vee \neg A) \vdash \neg A}}{\neg(A \vee \neg A) \vdash A \vee \neg A}}{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}}{\neg(A \vee \neg A) \vdash} \text{contraction}}{\neg(A \vee \neg A) \vdash} \text{contraction}} \text{contraction}$$

# The Excluded-Middle in Intuitionistic Logic

- ▶ is not provable. However, its **negation** is inconsistent.
- ▶ this suggests a scheme for a translation between int. and clas. logic:

$$\begin{array}{c}
 \frac{}{A \vdash A} \text{ax} \\
 \\
 \frac{}{\vdash A, \neg A} \neg R \\
 \\
 \frac{}{\vdash A \vee \neg A} \vee R
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{}{A \vdash A} \text{ax} \\
 \frac{A \vdash A}{A \vdash A \vee \neg A} \vee R1 \\
 \frac{A \vdash A \vee \neg A}{\neg(A \vee \neg A), A \vdash} \neg L \\
 \frac{\neg(A \vee \neg A), A \vdash}{\neg(A \vee \neg A) \vdash \neg A} \neg R \\
 \frac{\neg(A \vee \neg A) \vdash \neg A}{\neg(A \vee \neg A) \vdash A \vee \neg A} \vee R2 \\
 \frac{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}{\neg(A \vee \neg A) \vdash} \text{contr}
 \end{array}$$

- ▶ need:  $\neg\neg$  **everywhere** in  $\Delta$  (and  $\Gamma$ )
- ▶ the proof of the “negation of the excluded middle” requires **duplication** (contraction), which partly explain why we allow several formulæ on the rhs in LK.

# The Excluded-Middle in Intuitionistic Logic

- ▶ is not provable. However, its **negation** is inconsistent.
- ▶ this suggests a scheme for a translation between int. and clas. logic:

$$\begin{array}{c}
 \frac{}{A \vdash A} \text{ax} \\
 \\
 \frac{}{\vdash A, \neg A} \neg R \\
 \\
 \frac{}{\vdash A \vee \neg A} \vee R
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{}{A \vdash A} \text{ax} \\
 \frac{A \vdash A}{A \vdash A \vee \neg A} \vee R1 \\
 \frac{A \vdash A \vee \neg A}{\neg(A \vee \neg A), A \vdash} \neg L \\
 \frac{\neg(A \vee \neg A), A \vdash}{\neg(A \vee \neg A) \vdash \neg A} \neg R \\
 \frac{\neg(A \vee \neg A) \vdash \neg A}{\neg(A \vee \neg A) \vdash A \vee \neg A} \vee R2 \\
 \frac{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}{\neg(A \vee \neg A) \vdash} \neg L \\
 \text{contr}
 \end{array}$$

- ▶ given a classical proof  $\Gamma \vdash \Delta$ , **store**  $\Delta$  on the lhs, and translate:

$$\begin{array}{c}
 \text{Clas.} \\
 \text{rule } r \frac{\Gamma \vdash A_1, \Delta \quad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A, \Delta}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Int.} \\
 \text{rule } r \frac{\frac{\Gamma, \neg A_1, \neg \Delta \vdash}{\Gamma, \neg \Delta \vdash \neg \neg A_1} \neg R \quad \frac{\Gamma, \neg A_2, \neg \Delta \vdash}{\Gamma, \neg \Delta \vdash \neg \neg A_2} \neg R}{\frac{\Gamma, \neg \Delta \vdash A}{\Gamma, \neg \Delta, \neg A \vdash} \neg L}
 \end{array}$$

- ▶ need:  $\neg\neg$  **everywhere** in  $\Delta$  (and  $\Gamma$ )
- ▶ the proof of the “negation of the excluded middle” requires **duplication** (contraction), which partly explain why we allow several formulæ on the rhs in LK.



# Kolmogorov's Translation

Kolmogorov's  $\neg\neg$ -translation introduces  $\neg\neg$  everywhere:

$$\begin{aligned} B^{Ko} &= \neg\neg B && \text{(atoms)} \\ (B \wedge C)^{Ko} &= \neg\neg(B^{Ko} \wedge C^{Ko}) \\ (B \vee C)^{Ko} &= \neg\neg(B^{Ko} \vee C^{Ko}) \\ (B \Rightarrow C)^{Ko} &= \neg\neg(B^{Ko} \Rightarrow C^{Ko}) \\ (\forall xA)^{Ko} &= \neg\neg(\forall xA^{Ko}) \\ (\exists xA)^{Ko} &= \neg\neg(\exists xA^{Ko}) \end{aligned}$$

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^{Ko}, \lrcorner\Delta^{Ko} \vdash$  is provable in LJ.

## Antinegation

$\lrcorner$  is an **operator**, such that:

- ▶  $\lrcorner\neg A = A$ ;
- ▶  $\lrcorner B = \neg B$  otherwise.

# Light Kolmogorov's Translation

Moving negation from connectives to formulæ [Dowek& Werner]:

$$\begin{aligned} B^K &= B && \text{(atoms)} \\ (B \wedge C)^K &= (\neg\neg B^K \wedge \neg\neg C^K) \\ (B \vee C)^K &= (\neg\neg B^K \vee \neg\neg C^K) \\ (B \Rightarrow C)^K &= (\neg\neg B^K \Rightarrow \neg\neg C^K) \\ (\forall x A)^K &= \forall x \neg\neg A^K \\ (\exists x A)^K &= \exists x \neg\neg A^K \end{aligned}$$

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^K, \neg\neg\Delta^K \vdash$  is provable in LJ.

## Correspondence

$$A^{Ko} = \neg\neg A^K$$

# How does the Translation Work ?

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^K, \neg\Delta^K \vdash$  is provable in LJ.

**Proof:** Induction on the LK proof.  $\neg$  bounces. Example: rule  $\wedge_R$ .

$$\wedge_R \frac{\frac{\pi_1}{\Gamma \vdash A, \Delta} \quad \frac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\frac{\frac{\pi'_1}{\Gamma^K, \neg A^K, \neg\Delta^K \vdash} \quad \frac{\pi'_2}{\Gamma^K, \neg B^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg(\neg\neg A^K \wedge \neg\neg B^K), \neg\Delta^K \vdash} \wedge_R$$

# How does the Translation Work ?

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^K, \neg\Delta^K \vdash$  is provable in LJ.

**Proof:** Induction on the LK proof.  $\neg$  bounces. Example: rule  $\wedge_R$ .

$$\wedge_R \frac{\frac{\pi_1}{\Gamma \vdash A, \Delta} \quad \frac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\frac{\frac{\pi'_1}{\Gamma^K, \neg A^K, \neg\Delta^K \vdash} \quad \frac{\pi'_2}{\Gamma^K, \neg B^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg(\neg\neg A^K \wedge \neg\neg B^K), \neg\Delta^K \vdash} \wedge_R$$

# How does the Translation Work ?

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^K, \neg\Delta^K \vdash$  is provable in LJ.

**Proof:** Induction on the LK proof.  $\neg$  bounces. Example: rule  $\wedge_R$ .

$$\wedge_R \frac{\frac{\pi_1}{\Gamma \vdash A, \Delta} \quad \frac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_L \frac{\frac{\frac{\pi'_1}{\Gamma^K, \neg A^K, \neg\Delta^K \vdash} \quad \frac{\pi'_2}{\Gamma^K, \neg B^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg\Delta^K \vdash \neg\neg A^K \wedge \neg\neg B^K} \quad \wedge_R}{\Gamma^K, \neg(\neg\neg A^K \wedge \neg\neg B^K), \neg\Delta^K \vdash}$$

# How does the Translation Work ?

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^K, \neg\Delta^K \vdash$  is provable in LJ.

**Proof:** Induction on the LK proof.  $\neg$  bounces. Example: rule  $\wedge_R$ .

$$\wedge_R \frac{\frac{\pi_1}{\Gamma \vdash A, \Delta} \quad \frac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_R \frac{\frac{\frac{\pi'_1}{\Gamma^K, \neg A^K, \neg\Delta^K \vdash} \quad \frac{\pi'_2}{\Gamma^K, \neg B^K, \neg\Delta^K \vdash}}{\Gamma^K, \neg\Delta^K \vdash \neg\neg A^K} \quad \frac{\Gamma^K, \neg\Delta^K \vdash \neg\neg B^K}{\Gamma^K, \neg\Delta^K \vdash \neg\neg A^K \wedge \neg\neg B^K} \neg_R}{\Gamma^K, \neg(\neg\neg A^K \wedge \neg\neg B^K), \neg\Delta^K \vdash} \neg_L \wedge_R$$

# Are they morphisms ?

Consider Kolmogorov's translation:

- ▶ let:

$$\begin{aligned}B \wedge_c C &= \neg\neg(B \wedge_i C) \\B \vee_c C &= \neg\neg(B \vee_i C) \\B \Rightarrow_c C &= \neg\neg(B \Rightarrow_i C) \\ \forall_c xA &= \neg\neg(\forall_i xA) \\ \exists_c xA &= \neg\neg(\exists_i xA)\end{aligned}$$

- ▶ unfortunately:

$$\begin{aligned}B^{Ko} &= \neg\neg B && \text{(atoms)} \\(B \wedge C)^{Ko} &= B^{Ko} \wedge_c C^{Ko} \\(B \vee C)^{Ko} &= B^{Ko} \vee_c C^{Ko} \\(B \Rightarrow C)^{Ko} &= B^{Ko} \Rightarrow_c C^{Ko} \\(\forall xA)^{Ko} &= \forall_c xA^{Ko} \\(\exists xA)^{Ko} &= \exists_c xA^{Ko}\end{aligned}$$

- ▶ this is not a morphism.

# Are they morphisms ?

- ▶ No !

- ★ in the case of  $Ko$ :

$$B^{Ko} = \neg\neg B(\text{atoms})$$

- ★ in the case of  $K$  :

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^K, \neg\Delta^K \vdash$  is provable in LJ.

- ★ exercise: these negations are necessary (hint: consider the excluded-middle and its derivatives)

- ▶ can we be more clever ?

- ★ some **intuitionistic** right-rules are the same as **classical** right-rules. For instance,  $\wedge_R$ :

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

- ★ Translate them by themselves. Gödel-Gentzen translation.



# Gödel-Gentzen Translation

In this translation, disjunctions and existential quantifiers are replaced by a combination of negation and their De Morgan duals:

$$\begin{aligned}B^{gg} &= \neg\neg B \\(A \wedge B)^{gg} &= A^{gg} \wedge B^{gg} \\(A \vee B)^{gg} &= \neg(\neg A^{gg} \wedge \neg B^{gg}) \\(A \Rightarrow B)^{gg} &= A^{gg} \Rightarrow B^{gg} \\(\forall xA)^{gg} &= \forall xA^{gg} \\(\exists xA)^{gg} &= \neg\forall x\neg A^{gg}\end{aligned}$$

## Example of translation

$((A \vee B) \Rightarrow C)^{gg}$  is  $(\neg(\neg\neg\neg A \wedge \neg\neg\neg B)) \Rightarrow \neg\neg C$

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^{gg}, \lrcorner\Delta^{gg} \vdash$  is provable in LJ.

# Are they morphisms ?

▸ No !

★ in the case of  $Ko$ :

$$B^{Ko} = \neg\neg B(\text{atoms})$$

★ in the case of  $K$  :

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^K, \neg\Delta^K \vdash$  is provable in LJ.

★ exercise: show that those negations are necessary (hint: consider the excluded-middle and its derivatives)

▸ can we be more clever ?

★ some **intuitionistic** right-rules are the same as **classical** right-rules. For instance,  $\wedge_R$ :

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

★ Gödel-Getzen translation:

★ **is still not a morphism !**

▸ etc. for all the other known translations (Krivine, Kuroda, Frédéric Gilbert)

# How to make a morphism: an analysis

- ▶ Translation of, say,  $A \wedge B$ :

Kolmogorov	Light Kolmogorov
$\neg\neg(A^{Ko} \wedge B^{Ko})$	$(\neg\neg A^{Ko}) \wedge (\neg\neg B^{Ko})$

- ▶ Feature, **double-negation**:

Kolmogorov	Light Kolmogorov
<b>on top</b> of connectives	<b>inside</b> connectives

- ▶ Analysis, **problem** appearing in:

	Kolmogorov	Light Kolmogorov
Problem	atoms: $\neg\neg P$	statement: $\Gamma^K, \neg\Delta^K \vdash$
Solution	statement: $\Gamma^{Ko}, \lrcorner\Delta^{Ko} \vdash$	atoms: $P$

- ▶ Solution: **combine** them !

# Dowek's translation

$$\begin{aligned} B^D &= B & &= B & & \text{(atoms)} \\ (B \wedge C)^D &= B^D \wedge_c C^D & &= \neg\neg(\neg\neg B^D \wedge \neg\neg C^D) \\ (B \vee C)^D &= B^D \vee_c C^D & &= \neg\neg(\neg\neg B^D \vee \neg\neg C^D) \\ (B \Rightarrow C)^D &= B^D \Rightarrow_c C^D & &= \neg\neg(\neg\neg B^D \Rightarrow \neg\neg C^D) \\ (\forall x A)^D &= \forall_c x A^D & &= \neg\neg \forall x \neg\neg A^D \\ (\exists x A)^D &= \exists_c x A^D & &= \neg\neg \exists x \neg\neg A^D \end{aligned}$$

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^D, \lrcorner \Delta^D \vdash$  is provable in LJ.

## Corollary

Assume  $A$  is not atomic.  $\Gamma \vdash A$  is provable in LK iff  $\Gamma^D \vdash A^D$  is provable in LJ.

Proof:

- ▶  $\lrcorner \lrcorner A^D = A^D$  (except in the atomic case)  $\square$

# The Price to Pay

- ▶ heavy: for each connective, 6 negations.  $((A \vee B) \Rightarrow C)^D$  is  $\neg\neg(\neg\neg\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg C)$
- ▶ most of the time **useless**, except at the top and at the bottom of the formula
- ▶ remember Gödel-Gentzen's idea. Use De Morgan duals:

$$(A \vee B)^{gg} = \neg(\neg A^{gg} \vee \neg B^{gg})$$

- ▶ let us do the same, and divide by two the number of double negations.

# A Light Morphism

Remember De Morgan,

$$A \vee B = \neg(\neg A \wedge \neg B)$$

$$A \wedge B = \neg(\neg A \vee \neg B)$$

$$A \Rightarrow B = \neg A \vee B$$

$$\neg\neg A = A$$

$$\neg\forall x A = \exists x \neg A$$

$$\neg\exists x A = \forall x \neg A$$

# A Light Morphism

Remember De Morgan, and let

$$A \vee_c B = \neg(\neg A \wedge \neg B)$$

$$A \wedge_c B = \neg(\neg A \vee \neg B)$$

$$A \Rightarrow_c B = \neg(\neg\neg A \vee \neg B)$$

$$\neg_c A = \neg\neg\neg A$$

$$\forall_c xA = \neg\exists x\neg A$$

$$\exists_c xA = \neg\forall x\neg A$$

- ▶ this gives rise to a morphism,  $(.)^\circ$  together with:

$$\top_c = \neg\neg\top$$

$$\perp_c = \neg\neg\perp$$

- ▶ and we can prove the theorem:

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^\circ, \perp\Delta^\circ \vdash$  is provable in LJ.

# Some Cases

Proof by induction on the proof of  $\Gamma \vdash \Delta$ .

- ▶ last rule  $\vee_R$  on some  $A \vee B \in \Delta$ . Remember:  $\lrcorner(A \vee B)^\circ = \neg A^\circ \wedge \neg B^\circ$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

- ★  $A$  and  $B$  are atomic:  $\lrcorner A^\circ = \neg A$  and  $\lrcorner B^\circ = \neg B$ .

$$\frac{\Gamma^\circ, \neg A, \neg B, \lrcorner \Delta^\circ \vdash}{\Gamma^\circ, \neg A \wedge \neg B, \lrcorner \Delta^\circ \vdash}$$

- ★ if neither  $A$  and  $B$  are atomic, then  $A^\circ$  and  $B^\circ$  have a trailing  $\neg$ , and we remove it (bouncing):

$$\frac{\frac{\Gamma^\circ, \lrcorner A^\circ, \lrcorner B^\circ, \lrcorner \Delta^\circ \vdash}{\Gamma^\circ, \neg A^\circ, \neg B^\circ, \lrcorner \Delta^\circ \vdash}}{\Gamma^\circ, \neg A^\circ \wedge \neg B^\circ, \lrcorner \Delta^\circ \vdash} (\neg_R, \neg_L) \times 2$$

- ★ mixed case: mixed strategy.



# Conclusion

- ▶ logic with two kinds of connectives:  $\vee_i$  and  $\vee_c$

$$\vee_{R1} \frac{\Gamma \vdash A}{\Gamma \vdash A \vee_i B} \quad \vee_{R2} \frac{\Gamma \vdash B}{\Gamma \vdash A \vee_i B} \quad \vee_{cR} \frac{\Gamma, \neg A, \neg B \vdash}{\Gamma \vdash A \vee_c B}$$

- ▶ and we have:

If  $\Gamma, \Delta, A$  contain only classical connectives,  $A$  non atomic, then  $\Gamma \vdash A$  in LK iff  $\Gamma \vdash A$ . As well,  $\Gamma \vdash \Delta$  in LK iff  $\Gamma, \lrcorner \Delta \vdash$ .

- ▶ Getting lighter morphisms:
  - ★ from  $\neg_c A = \neg \neg \neg A$  to  $\neg_c A = \neg A$  ?
  - ★ from  $A \Rightarrow_c B = \neg(\neg \neg A \vee \neg B)$  to  $A \Rightarrow_c B = \neg(A \vee \neg B)$  ?
  - ★ we cannot always maintain the invariant  $\Gamma, \lrcorner \Delta \vdash$ .
  - ★ **Focusing** in LK to the rescue.

# Lighter Morphisms

Remember De Morgan, and let:

$$A \vee_c B = \neg(\neg A \wedge \neg B)$$

$$A \wedge_c B = \neg(\neg A \vee \neg B)$$

$$A \Rightarrow_c B = \neg(\neg\neg A \vee \neg B)$$

$$\neg_c A = \neg\neg\neg A$$

$$\forall_c xA = \neg\exists x\neg A$$

$$\exists_c xA = \neg\forall x\neg A$$

# Lighter Morphisms

Remember De Morgan, and let:

$$A \vee_c B = \neg(\neg A \wedge \neg B)$$

$$A \wedge_c B = \neg(\neg A \vee \neg B)$$

$$A \Rightarrow_c B = \neg(\neg\neg A \vee \neg B)$$

$$\neg_c A = \neg A$$

$$\forall_c xA = \neg\exists x\neg A$$

$$\exists_c xA = \neg\forall x\neg A$$

- ▶ this gives rise to a morphism,  $(.)^\circ$  together with:

$$\top_c = \top$$

$$\perp_c = \perp$$

- ▶ and we can prove the theorem:

## Theorem

$\Gamma \vdash \Delta$  is provable in LK iff  $\Gamma^\circ, \lrcorner\Delta^\circ \vdash$  is provable in LJ.

# Dismissing Cuts ?

The cut rule:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta} \text{ cut}$$

- ▶ connects rhs and lhs
- ▶ **dual** to the axiom rule:

$$\overline{\Gamma, A \vdash A, \Delta} \text{ axiom}$$

- ▶ is **admissible/eliminable** (Gentzen)

If we do not have connections, then we can differentiate rhs and lhs translations:

$$\begin{array}{ll} A \vee_c^l B = A \vee B & A \vee_c^r B = \neg(\neg A \wedge \neg B) \\ A \wedge_c^l B = A \wedge B & A \wedge_c^r B = \neg(\neg A \vee \neg B) \\ A \Rightarrow_c^l B = A \Rightarrow B & A \Rightarrow_c^r B = \neg(\neg\neg A \vee \neg B) \\ \neg_c^l A = \neg A & \neg_c^r A = \neg\neg\neg A \\ \forall_c^l xA = \forall xA & \forall_c^r xA = \neg\exists x\neg A \\ \exists_c^l xA = \exists xA & \exists_c^r xA = \neg\forall x\neg A \end{array}$$

# A Focus on LK $\rightarrow$ LJ

- less negations imposes more discipline. Example:

$$\begin{array}{c}
 \frac{\pi_1}{\Gamma \vdash A, \Delta} \quad \frac{\pi_2}{\Gamma \vdash B, \Delta} \\
 \hline
 \Gamma \vdash A \wedge B, \Delta \\
 \wedge_R
 \end{array}
 \quad \text{becomes} \quad
 \begin{array}{c}
 \frac{\pi'_1}{\Gamma^l, \lrcorner A^r, \lrcorner \Delta^r \vdash} \quad \frac{\pi'_2}{\Gamma^l, \lrcorner B^r, \lrcorner \Delta^r \vdash} \\
 \hline
 \Gamma^l, \lrcorner \Delta^r \vdash A^r \quad \Gamma^l, \lrcorner \Delta^r \vdash B^r \\
 \hline
 \lrcorner_L \frac{\Gamma^l, \lrcorner \Delta^r \vdash A^r \wedge B^r}{\Gamma^l, \lrcorner(A^r \wedge B^r), \lrcorner \Delta^r \vdash} \\
 \wedge_R
 \end{array}$$

- when  $A^r$  introduces negations ( $\exists, \forall, \lrcorner$  and atomic cases)  $??$  can be  $\lrcorner_R$  due to the behavior of  $\lrcorner A^r$
- otherwise  $A^r$  remains of the rhs in the LJ proof.

# A Focus on LK $\rightarrow$ LJ

- less negations imposes more discipline. Example:

$$\begin{array}{c}
 \frac{\pi_1}{\Gamma \vdash A, \Delta} \quad \frac{\pi_2}{\Gamma \vdash B, \Delta} \\
 \hline
 \Gamma \vdash A \wedge B, \Delta \quad \text{becomes} \quad \frac{\frac{\frac{\pi'_1}{\Gamma^l, \lrcorner A^r, \lrcorner \Delta^r \vdash} \dots}{\Gamma^l, \lrcorner \Delta^r \vdash A^r} \quad \frac{\frac{\pi'_2}{\Gamma^l, \lrcorner B^r, \lrcorner \Delta^r \vdash} \dots}{\Gamma^l, \lrcorner \Delta^r \vdash B^r}}{\Gamma^l, \lrcorner \Delta^r \vdash A^r \wedge B^r} \quad \lrcorner_L \\
 \hline
 \Gamma^l, \lrcorner(A^r \wedge B^r), \lrcorner \Delta^r \vdash \quad \text{becomes} \quad \frac{\Gamma^l, \lrcorner(A^r \wedge B^r), \lrcorner \Delta^r \vdash}{\Gamma^l, \lrcorner(A^r \wedge B^r), \lrcorner \Delta^r \vdash} \quad \lrcorner_R
 \end{array}$$

- when  $A^r$  introduces negations ( $\exists, \forall, \lrcorner$  and atomic cases)  $??$  can be  $\lrcorner_R$  due to the behavior of  $\lrcorner A^r$
- otherwise  $A^r$  remains of the rhs in the LJ proof.
- the next rule in  $\pi_1$  and  $\pi_2$  **must** be on  $A$  (resp.  $B$ ). How ?

# A Focus on LK $\rightarrow$ LJ

- less negations imposes more discipline. Example:

$$\begin{array}{c}
 \frac{\pi_1}{\Gamma \vdash A, \Delta} \quad \frac{\pi_2}{\Gamma \vdash B, \Delta} \\
 \hline
 \Gamma \vdash A \wedge B, \Delta \quad \text{becomes} \quad \frac{\frac{\frac{\pi'_1}{\Gamma^l, \lrcorner A^r, \lrcorner \Delta^r \vdash} \quad \frac{\pi'_2}{\Gamma^l, \lrcorner B^r, \lrcorner \Delta^r \vdash}}{\Gamma^l, \lrcorner \Delta^r \vdash A^r \wedge B^r} \quad \lrcorner L}{\Gamma^l, \lrcorner (A^r \wedge B^r), \lrcorner \Delta^r \vdash} \quad \wedge R
 \end{array}$$

- when  $A^r$  introduces negations ( $\exists, \forall, \lrcorner$  and atomic cases)  $??$  can be  $\lrcorner R$  due to the behavior of  $\lrcorner A^r$
- otherwise  $A^r$  remains of the rhs in the LJ proof.
- the next rule in  $\pi_1$  and  $\pi_2$  **must** be on  $A$  (resp.  $B$ ). How ?
- use Kleene's inversion lemma
- or ... this is exactly what focusing is about !

# A Focused Classical Sequent Calculus

## Sequent with focus

A focused sequent  $\Gamma \vdash A; \Delta$  has three parts:

- ▶  $\Gamma$  and  $\Delta$
- ▶  $A$ , the (possibly empty) **stoup formula**

$$\Gamma \vdash \underbrace{\cdot}_{\text{stoup}}; \Delta$$

- ▶ when the stoup is not empty, the next rule must apply on its formula,
- ▶ under some conditions, it is possible to move/remove a formula in/from the stoup.



# A Focused Sequent Calculus

$$\frac{}{\Gamma, A \vdash \cdot ; A, \Delta} \text{ax}$$

$$\frac{\Gamma, A, B \vdash \cdot ; \Delta}{\Gamma, A \wedge B \vdash \cdot ; \Delta} \wedge_L$$

$$\frac{\Gamma \vdash A ; \Delta \quad \Gamma \vdash B ; \Delta}{\Gamma \vdash A \wedge B ; \Delta} \wedge_R$$

$$\frac{\Gamma, A \vdash \cdot ; \Delta \quad \Gamma, B \vdash \cdot ; \Delta}{\Gamma, A \vee B \vdash \cdot ; \Delta} \vee_L$$

$$\frac{\Gamma \vdash \cdot ; A, B, \Delta}{\Gamma \vdash \cdot ; A \vee B, \Delta} \vee_R$$

$$\frac{\Gamma \vdash A ; \Delta \quad \Gamma, B \vdash \cdot ; \Delta}{\Gamma, A \Rightarrow B \vdash \cdot ; \Delta} \Rightarrow_L$$

$$\frac{\Gamma, A \vdash B ; \Delta}{\Gamma \vdash A \Rightarrow B ; \Delta} \Rightarrow_R$$

$$\frac{\Gamma, A[c/x] \vdash \cdot ; \Delta}{\Gamma, \exists x A \vdash \cdot ; \Delta} \exists_L$$

$$\frac{\Gamma \vdash \cdot ; A[t/x], \Delta}{\Gamma \vdash \cdot ; \exists x A, \Delta} \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \cdot ; \Delta}{\Gamma, \forall x A \vdash \cdot ; \Delta} \forall_L$$

$$\frac{\Gamma \vdash A[c/x] ; \Delta}{\Gamma \vdash \forall x A ; \Delta} \forall_R$$

$$\frac{\Gamma \vdash A ; \Delta}{\Gamma \vdash \cdot ; A, \Delta} \text{focus}$$

$$\frac{\Gamma \vdash \cdot ; A, \Delta}{\Gamma \vdash A ; \Delta} \text{release}$$

# A Focused Sequent Calculus

$$\frac{\Gamma \vdash A ; \Delta}{\Gamma \vdash . ; A, \Delta} \text{ focus} \quad \frac{\Gamma \vdash . ; A, \Delta}{\Gamma \vdash A ; \Delta} \text{ release}$$

Characteristics:

- ▶ in **release**,  $A$  is either atomic or of the form  $\exists xB, B \vee C$  or  $\neg B$ ;
- ▶ in **focus**, the converse holds:  $A$  must not be atomic, nor of the form  $\exists xB, B \vee C$  nor  $\neg B$ .
- ▶ the *synchronous* (outside the stoup) right-rules are  $\exists_R, \neg_R, \vee_R$  and (atomic) axiom: the exact places where  $\{.\}^r$  introduces negation

## Theorem

If  $\Gamma \vdash \Delta$  is provable in LK then  $\Gamma \vdash . ; \Delta$  is provable.

Proof: use Kleene's inversion lemma (holds for all connectives/quantifiers, except  $\exists_R$  and  $\forall_L$ ).

# Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash A ; \Delta}{\Gamma \vdash . ; A, \Delta} \text{focus} \quad \frac{\Gamma \vdash . ; A, \Delta}{\Gamma \vdash A ; \Delta} \text{release}$$

## Theorem

If  $\Gamma \vdash A ; \Delta$  in focused LK, then  $\Gamma^l, \lrcorner \Delta^r \vdash A^r$  in LJ

- ▶ **release** is translated by the  $\neg_R$  rule
- ▶ **focus** is translated by the  $\neg_L$  rule

# Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash A ; \Delta}{\Gamma \vdash \cdot ; A, \Delta} \text{focus} \quad \frac{\Gamma \vdash \cdot ; A, \Delta}{\Gamma \vdash A ; \Delta} \text{release}$$

## Theorem

If  $\Gamma \vdash A ; \Delta$  in focused LK, then  $\Gamma^l, \lrcorner \Delta^r \vdash A^r$  in LJ

- ▶ **release** is translated by the  $\neg_R$  rule
- ▶ **focus** is translated by the  $\neg_L$  rule
- ▶  $\lrcorner \Delta^r$  removes the trailing negation on  $\exists^n (\neg \forall \neg)$ ,  $\forall^r (\neg \wedge \neg)$ ,  $\neg^r (\neg)$  and atoms  $(\neg \neg)$
- ▶ what a surprise: focus is forbidden on them, so rule on the lhs:

LK rule	$\exists_R$	$\neg_R$	$\forall_R$	ax.
LJ rule	$\forall_L$	nop	$\wedge_L$	$\neg_L + \text{ax.}$

- ▶ see the paper.