From physics to interrupt handlers: the real to float step

Vivien Maisonneuve

Presentation at Deducteam

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Different levels of description

In control engineering, work on different levels to design and build a control system:

→

- Format/high-level aspects: system conception, modeling, possibly proof.
- Concrete/low-level aspects: creation of an object implementing the system.

  Quadricopter, DRONE Project, MINES ParisTech & ECP.

Formal aspect

model ⟶ object

System definition:

- Inputs: sensors [accelerometer, sonar...] + references [operator instructions].
  Outputs: actions to act on environment [rotors rotation speed].
- Modeling in the form of equations to express relations between inputs and outputs: transfer functions or differential equations.

Formal aspect

model object

System definition:

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  Outputs: actions to act on environment [rotors rotation speed].
- Modeling in the form of equations to express relations between inputs and outputs: transfer functions or differential equations.

System requirements:

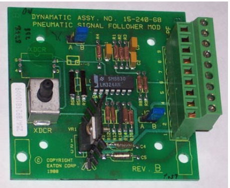
- Stability conditions [bounded rotation speed].
- Pursuit of reference input [try to reach the ordered position].
- Perturbation rejection [wind].

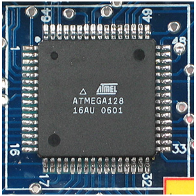
Concrete aspect

model ⟶ object

Creation of a real object implementing the system.

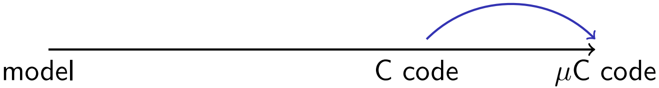
- Electronic circuit that physically computes the transfer function.
- With a **microcontroller**: a small system with processor, memory, I/O devices, that runs a program implementing the transfer function.





[ATMEGA128
Frequency: 16 MHz
RAM: 4 KB
Prog. mem.: 128 KB]

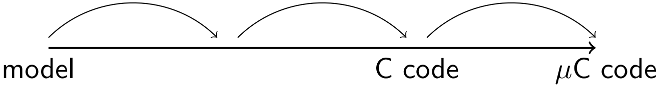
Semantic gap



Antagonism:

- Abstract, mathematical model.
- Microcontroller code: program written in C, then compiled.
  Long (thousands of LoC), low-level (elementary operations, hardware management, interruptions).

Semantic gap



Antagonism:

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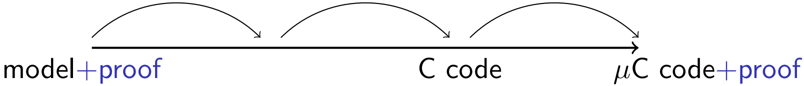
Series of transformations to go from abstract model to microcontroller code.

Semantic gap



Antagonism:

- Abstract, mathematical model.
- Microcontroller code: program written in C, then compiled.

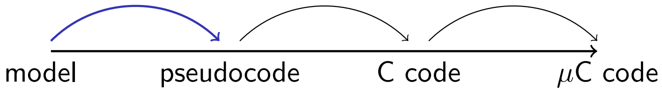
Series of transformations to go from abstract model to microcontroller code.

Problem of proof transposition: Considering a model with correction proofs [stability], how to transpose down these proofs at the code level?

Interest: formally check the code, not only the model.

Difficulties: semantic gap, non-equivalent transformations ($\Rightarrow$ proofs must be checked).

Control-theoretical aspects



Produce a pseudocode from the abstract model:

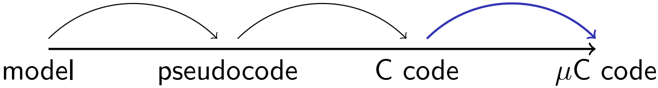
- Solve the model differential equation, get a transfer function.
  (Laplace transform/Z transform, initial conditions problem.)
- If continuous-time model, discretization.
  (Problems with sampling, execution times.)

while transposing the proof.

Usual problems in control engineering.

Once done, discrete-time system with a loop on the transfer function $\Rightarrow$ pseudocode [in MATLAB]. Proof: invariants on this code.

Compilation aspects



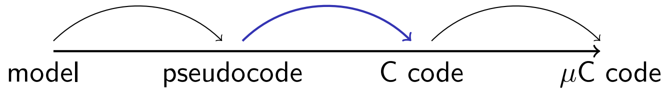
At the other end: compilation of C code to machine code.
Risks of error:

- Important changes in the code: elementary operations, management of registers and of memory stack, instruction jumps.
- Possible optimizations.

Solutions:

- "Existing C compilers are reliable enough."
- Proof-preserving compilation [Barthe].
- Certified compilation [CompCert].

What's between?

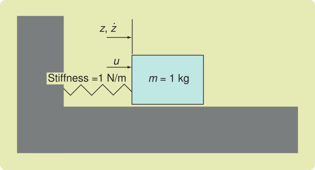


Opener question. Several challenges:

1. ~~High level mathematical operations~~ $\leadsto$ series of elementary instructions [matrices, sinus].
2. ~~Real values~~ $\leadsto$ machine words with limited precision.
3. On a microcontroller, data/events acquisition raises interruptions (real-time answer, energy consumption) $\Rightarrow$ particular code structure.

Example system

Very simple, linear invariant system.



Pseudocode:

```
Ac = [0.4990, -0.0500; 0.0100, 1.0000];
Bc = [1;0];
Cc = [564.48, 0];
Dc = -1280;

xc = zeros(2,1);

receive(y,2); receive(yd,2);
while 1
  yc = max(min(y - yd,1),-1);
  u = Cc*xc + Dc*yc;
  xc = Ac*xc + Bc*yc;
  send(u,1);
  receive(y,2);
  receive(yd,2);
end
```

state matrix (matrice de dynamique)
input matrix (matrice de commande)
output matrix (matrice d'observation)
feedthrough matrix (matrice d'action directe)

$x_c = \begin{pmatrix} x_{c_1} \\ x_{c_2} \end{pmatrix} \in \mathbb{R}^2$: controller state

$y \in \mathbb{R}$: reference input; $y_d \in \mathbb{R}$: real position

$y_c \in [-1, 1]$: bounded gap

$u \in \mathbb{R}$: action to be performed

Lyapunov stability

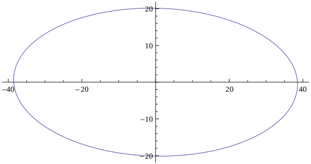
Lyapunov stability: all reachable states x_c start near an equilibrium point x_e and stay in a neighborhood V of x_e forever.

V found solving a Lyapunov equation. On linear systems, V is generally an ellipsoid.

Here, show that $x_c = \begin{pmatrix} x_{c_1} \\ x_{c_2} \end{pmatrix}$ belongs to the ellipse:

$$\mathcal{E}_P = \{x \in \mathbb{R}^2 \mid x^T \cdot P \cdot x \leq 1\}, \qquad P = 10^{-3} \begin{pmatrix} 0,6742 & 0,0428 \\ 0,0428 & 2,4651 \end{pmatrix}.$$

$$x_c \in \mathcal{E}_P \iff 0.6742 x_{c_1}^2 + 0.0856 x_{c_1} x_{c_2} + 2.4651 x_{c_2}^2 \leq 1000.$$



Stability proof

```
xc = zeros(2,1);
```
$x_c \in \mathcal{E}_P$
```
receive(y,2); receive(yd,2);
```
$x_c \in \mathcal{E}_P$
```
while 1
```
$x_c \in \mathcal{E}_P$
```
  yc = max(min(y - yd,1),-1);
```
$x_c \in \mathcal{E}_P, \quad y_c^2 \leq 1$

$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu} \quad | \quad Q_\mu = \begin{pmatrix} \mu P & 0_{2\times 1} \\ 0_{1\times 2} & 1-\mu \end{pmatrix}, \mu = 0.9991$
```
  u = Cc*xc + Dc*yc;
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$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$
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  xc = Ac*xc + Bc*yc;
```
$x_c \in \mathcal{E}_{\tilde{P}} \quad | \quad \tilde{P} = \left[\begin{pmatrix} A_c & B_c \end{pmatrix} \cdot Q_\mu^{-1} \cdot \begin{pmatrix} A_c & B_c \end{pmatrix}^T \right]^{-1}$
```
  send(u,1);
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$x_c \in \mathcal{E}_{\tilde{P}}$
```
  receive(y,2);
```
$x_c \in \mathcal{E}_{\tilde{P}}$
```
  receive(yd,2);
```
$x_c \in \mathcal{E}_{\tilde{P}}$

$x_c \in \mathcal{E}_P$
```
end
```

Proof given as code invariants

Implication (weakening) if two consecutive invariants.

Trivial, or easy to check with matrix computations.

Last implication closes the loop. Its validity relies on parameters A_c, B_c, C_c, D_c, μ: numerical verification needed.

Digression: with C instructions

~~High level mathematical operations~~ $\rightsquigarrow$ series of scalar elementary instructions.

Here, matrix operations are expanded: the instruction

$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$

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becomes:

$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$

```
xb[0] = xc[0];
xb[1] = xc[1];
xc[0] = Ac[0][0]*xb[0]+Ac[0][1]*xb[1]+yc;
xc[1] = Ac[1][0]*xb[0]+Ac[1][1]*xb[1];
```

xb: buffer

???

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xb: buffer

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Same computation: output invariant can be found [Feron].

Numerical precision problems

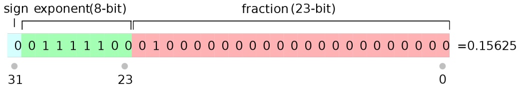
To produce C code: ~~real numbers~~ $\rightsquigarrow$ binary finite-length machine words (32 b. or 64 b.).

$\Rightarrow$ Loss in accuracy, two consequences:

1. Constant values are slightly altered.
2. Rounding errors during computations.

Machine representation of real numbers

1. Floating point: IEEE 754.
   Not usual on microcontrollers.



$$\text{number} = \text{sign} \times 2^{\text{exponent}+\text{cst. offset}} \times \text{fraction}$$

   Correct rounding for base operations: +, −, ∗, /.
   ⇒ If [bounds on] operands are known, not special, far enough from extremal values, then rounding error is bounded for +, −, ∗ (not /).

2. Fixed point.
   If operands are not special, far enough from extremal values, then rounding error is bounded for +, −, ∗.

3. Two integers.

Machine representation of real numbers

1. Floating point.
2. Fixed point.
3. Two integers. Rational representation: numerator, denominator.
   - Base behavior: +, −, ∗, / follow rational definition + fraction simplification:

     $$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \text{simpl}\left(\frac{p_1 q_2 + p_2 q_1}{q_1 q_2}\right), \text{ etc.}$$

     No rounding error.
     Problem: numerator value can easily exceed integer bounds.
   - Approximated behavior to ensure bounded numerator.

Alteration of constants

With IEEE 754, 32 bits, constants

```
Ac = [0.4990, -0.0500; 0.0100, 1.0000];
Bc = [1;0];
Cc = [564.48, 0];
Dc = -1280;
```

become

```
Ac ≈ [0.49900001287460327 , -0.05000000074505806;
      0.009999999776482582,  1.0000];
Bc ≈ [1;0];
Cc ≈ [564.47998046875, 0];
Dc ≈ -1280;
```

Effect on proof

```
xc = zeros(2,1);
```
$x_c \in \mathcal{E}_P$
```
receive(y,2); receive(yd,2);
```
$x_c \in \mathcal{E}_P$
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```
$x_c \in \mathcal{E}_P$
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  u = Cc*xc + Dc*yc;
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$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$
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  xc = Ac*xc + Bc*yc;
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```
  send(u,1);
```
$x_c \in \mathcal{E}_{\tilde{P}}$
```
  receive(y,2);
```
$x_c \in \mathcal{E}_{\tilde{P}}$
```
  receive(yd,2);
```
$x_c \in \mathcal{E}_{\tilde{P}}$

$x_c \in \mathcal{E}_P$
```
end
```

Rest of the code and proof sketch unchanged.

$\tilde{P}$ depends on A_c, B_c, C_c, D_c, is altered.

$\Rightarrow$ Last implication to be checked, might be wrong.

Rounding errors

With real numbers, the implication

$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$

```
xc = Ac*xc + Bc*yc;
```

$x_c \in \mathcal{E}_{\tilde{P}} \quad | \quad \tilde{P} = \left[\begin{pmatrix} A_c & B_c \end{pmatrix} \cdot Q_\mu^{-1} \cdot \begin{pmatrix} A_c & B_c \end{pmatrix}^T \right]^{-1}$

holds.

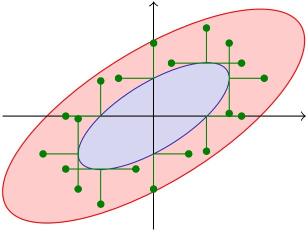
With floats, + and $*$ introduce rounding errors.

As x_c, y_c belong to an ellipsoid, they are bounded so the rounding error on x_c can be bounded by (e_1, e_2).

Super-ellipsoid

Let $\mathcal{E}_{\tilde{R}} \supset \mathcal{E}_{\tilde{P}}$ an ellipse s.t.

$$\forall x_c \in \mathcal{E}_{\tilde{P}},\ \forall x'_c \in \mathbb{R}^2,\ |x'_{c_1} - x_{c_1}| \leq e_1 \wedge |x'_{c_2} - x_{c_2}| \leq e_2 \Longrightarrow x'_c \in \mathcal{E}_{\tilde{R}} \quad (*)$$



Then:

$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$

```
xc = Ac*xc + Bc*yc;
```

$x_c \in \mathcal{E}_{\tilde{R}}$

$\mathcal{E}_{\tilde{R}}$ can be the smallest magnification of $\mathcal{E}_{\tilde{P}}$ s.t. $(*)$ holds.

Can be computed, whatever number of dimensions.

Effect on proof

```
xc = zeros(2,1);
```
$x_c \in \mathcal{E}_P$
```
receive(y,2); receive(yd,2);
```
$x_c \in \mathcal{E}_P$
```
while 1
```
$x_c \in \mathcal{E}_P$
```
  yc = max(min(y - yd,1),-1);
```
$x_c \in \mathcal{E}_P, \quad y_c^2 \leq 1$

$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu} \quad | \quad Q_\mu = \begin{pmatrix} \mu P & 0_{2\times 1} \\ 0_{1\times 2} & 1-\mu \end{pmatrix}, \mu = 0.9991$
```
  u = Cc*xc + Dc*yc;
```
$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$
```
  xc = Ac*xc + Bc*yc;
```
$x_c \in \mathcal{E}_{\tilde{R}}$
```
  send(u,1);
```
$x_c \in \mathcal{E}_{\tilde{R}}$
```
  receive(y,2);
```
$x_c \in \mathcal{E}_{\tilde{R}}$
```
  receive(yd,2);
```
$x_c \in \mathcal{E}_{\tilde{R}}$

$x_c \in \mathcal{E}_P$
```
end
```

Replace $\mathcal{E}_{\tilde{P}}$ by $\mathcal{E}_{\tilde{R}}$ in proof sketch.

Last implication to be checked, might be wrong.

Here it works: system stable with floats.

Other functions

Elementary operations +, $*$ are sufficient for linear, invariant systems.
The method applies if the proof sketch fits: no tight assumptions, complex operations on weakened invariants.

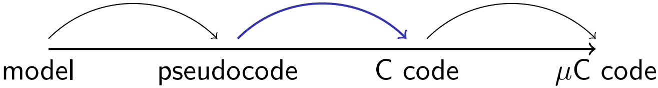
1-var, differentiable, periodic functions can be computed

- with an abacus and a polyhedral interpolation function
- wth a polyhedral approximation

with a boounded error (sin, cos).

Idem for 1-var, differentiable functions restricted to a finite range. OK if proof ensures the operand is bounded to the range.

Proof checking on C code



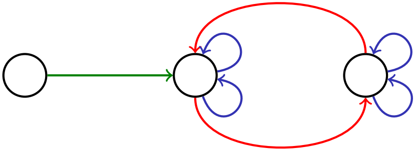
Transformations from pseudocode to C code are not equivalences.
$\Rightarrow$ The transposed proof sketch on C code might be false.
$\Rightarrow$ Check the C code invariants with an analyzer.

Attempt with PIPS.

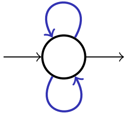
Interrupt handlers

```
SIGNAL (SIG_INPUT_CAPTURE3)
{
  ...
}
SIGNAL (SIG_SPI)
{
  ...
}
...
int main()
{
  initialize();
  enable_interrupts();
  while (1)
  {
    switch (state)
    {
      ...
    }
  }
  return 0;
}
```

Specific aspect of the code with interrupt handler: initialization followed by main loop, that can be interrupted at any time by signals.



Problem: structures with parallel loops difficult to analyze.



Polyhedral analysis

PIPS performs polyhedral analysis:
invariants = system of (in)equalities on the program variables (polyhedron). Good balance accuracy/complexity.

Usually, iterative approach: direct invariant propagation on control points, widening on cycles [Cousot-Halbwachs].

PIPS approach:

1. Abstract each instruction by a transfer relation (transformer), bottom to top. Links values before and after the instruction.

$$\texttt{x += y;} \quad \rightsquigarrow \quad \{x' = x + y \wedge y' = y\}$$

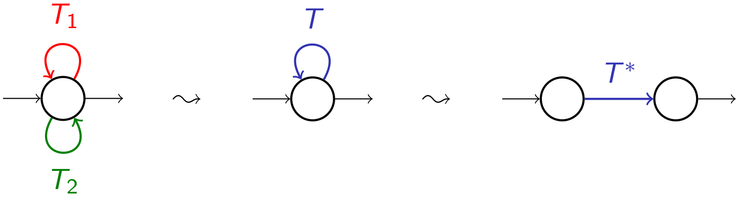
2. Invariant propagation on control points, using transformers.

Problem of parallel loops

When confronted to the code

```
while (rand()) {
  if (rand()) {c1}
  else {c2}
}
```

PIPS computes transformers T_1, T_2 associated to codes c_1, c_2,
then $T = T_1 \sqcup T_2$ the tranformer of the whole loop body,
then T^* the transformer corresponding to the loop.



Problem: loss in accuracy with $\sqcup$ and $*$, amplified when combined.
Too imprecise for many systems.

Different approaches

1. Refine transformers with invariants.
   Usual analysis with transformers then invariants.
   Then, restrict every transformer with its entry point invariant.
   Recompute invariants with new transformers.

   Does not converge in general.
   Rarely suited.
2. Delay convex hull.
3. Restructure the program.

Different approaches

1. Refine transformers with invariants.
2. Delay convex hull.
   Do not directly compute $T = T_1 \sqcup T_2$, then T^*.
   Instead, keep track of the list $[T_1, T_2]$ of involved transformers.
   Later, to propagate invariant P, do not compute

   $$P' = T^*(P)$$

   but instead:

   $$P' = \text{Comb}(\{T_1, T_2\}, P)$$

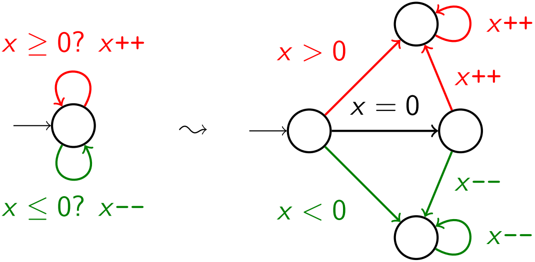
   with

   $$\text{Comb}(\{T_1, T_2\}, P) = P \sqcup T_1(P) \sqcup T_2(P) \sqcup T_1 \circ T_2(P) \sqcup T_2 \circ T_1(P) \\ \sqcup {T_1}^+(P) \sqcup {T_2}^+(P) \sqcup {T_1}^+ \circ T_2 \circ T^*(P) \sqcup {T_2}^+ \circ T_1 \circ T^*(P)$$

3. Restructure the program.

Different approaches

1. Refine transformers with invariants.
2. Delay convex hull.
3. Restructure the program.
   Transform into an equivalent program, easier to analyze.
   Idea: limit number of parallel loops by splitting control points according to loop guards.



Crucial point: choice of splitting partition. Manual or guided by a heuristic.

Different approaches

1. Refine transformers with invariants.
2. Delay convex hull.
3. Restructure the program.
   Best results: on 73 test cases,
   28 $\rightarrow$ 63 with PIPS, 47 $\rightarrow$ 70 with ASPIC [Gonnord].
   Equivalence certified with Coq.
   [NSAD'11].

Different approaches can be used simultaneously.
Work in progress.

From physics to interrupt handlers: the real to float step

Vivien Maisonneuve

Presentation at Deducteam

November 20, 2012