Eliminating Cuts in $hoI(\mathscr{C})$
Extended Abstract

Olivier Hermant
MINES ParisTech
PSL University

Jim Lipton
Dept. of Mathematics and Computer Science
Wesleyan University

We discuss the problem of cut elimination in an intuitionistic version of Church's Type Theory with constraints, a problem that arises in considering executable fragments suitable for logic programming.

The formal system $hoI(\mathscr{C})$, based on adding Saraswat constraints [13] to an intuitionistic formulation of Church's Theory of Types, was introduced in [9] in order to provide a framework for studying higher-order logic programming with constraints. To mediate between the constraint system and the logic, the following rule is introduced (where $\vdash_{\mathscr{C}}$ denotes entailment in the constraint system):

$$\frac{\Gamma \vdash_{\mathscr{C}} F_1^o \approx F_2^o}{\Delta, \Gamma, F_1 \ \vdash \ F_2}\ \text{Eq.}$$

The aim was to consider various uniform fragments of this logic with restricted proof search along the lines of λProlog [11] suitable for logic programming execution, as well as defining a semantics and proving soundness and completeness. Our perspective is shaped by the slogan of [9], "*constraints are generalized terms*".

This can be seen in the way constraints allow a more liberal notion of term. Even without the introduction of a theory as a constraint system (like the real numbers), we can express more terms with the help of high-order constraints. For instance, "a generic Church numeral" f, can be described as

$$\lambda a.(f\ a\ \lambda x.x) \approx \lambda a.a.$$

However the crucial question of cut elimination, without which logic programming becomes impracticable, remains unsettled. The problem is thorny since we can easily show it fails for arbitrary constraint systems. A counterexample to (the termination of) reductive cut elimination is easily found assuming only that the constraint system is able to derive a simple equality of predicates of the form $\vdash_{\mathscr{C}} B \approx (B \supset A)$ (B being an atomic predicate). Using the formalism of [9], the proof is:

$$\dfrac{\text{cut}^*\ \dfrac{\dfrac{\overset{(1)}{B \vdash A}}{\vdash B \supset A} \quad \text{Eq}\ \dfrac{\vdash_{\mathscr{C}} (B \supset A) \approx B}{B \supset A \vdash B}}{\vdash B} \qquad \dfrac{\text{Eq}\ \dfrac{\vdash_{\mathscr{C}} (B \supset A) \approx B}{B \supset A \vdash B} \quad \dfrac{\dfrac{}{B \vdash B} \quad \dfrac{}{A \vdash A}}{B, B \supset A \vdash A}}{B \vdash A}\ \text{cut}^*}{\vdash A}\ \text{cut}$$

where the proof-tree (1) is a copy of the right branch of the initial cut rule, and the two cut* rules are unavoidable and uneliminable equality cuts [6].

Let us now assume that A is derivable without cut. The cut on B in the above proof is still not eliminable mechanically, because the cut-free proof of A cannot be produced from the above proof. It

Submitted to:
TEASE 2020

has to be invented. So cut elimination fails, while the cut rule is admissible, and this question is bound to the provability of A, which is undecidable [2].

This example is realistic and could easily come from defining the constraint system with rewriting, in a Deduction-modulo-theory fashion [3], where such counterexamples do exist [4, 5]. For such systems, it is known that cut admissibility is more general than proof normalization, that both properties depend on the rewrite system, without any complete universal syntactic criterion, as they are undecidable [2]. Through an expression of rewriting as a constraint system, all those difficulties transfer automatically to $hoI(\mathscr{C})$.

Thus adding constraints to higher-order logic for logic programming purposes cannot be done indiscriminately, and the aim of the research presented here is to find sensible conditions on a constraint system that will allow cut elimination. The uniformity requirement developed in [9] is an effort in this direction, but it is not clear whether this is sufficient to ensure cut elimination.

Restrictions on the Logic

We start with a drastic restriction of $hoI(\mathscr{C})$ to first-order quantification, allowing constraints to hold only at term-level, in effect yielding to a first-order logic with constraints. Following the slogan mentioned above, namely that constraints are generalized terms, we build models where the domain of interpretation, instead of being terms, is built out of constraints themselves.

This leads us to the point of view that constraints are in fact generalized (and global) *substitutions*. Equipped with this, we build Heyting-like algebras as discussed below and show soundness and completeness theorems through the Lindenbaum construction, which elements are classes of provably equivalent formulas.

We then focus on a restricted instance of the constraint system with higher order logic, which yields an instance of $hoI(\mathscr{C})$ that is essentially equivalent to intuitionistic Church's Theory of Types. As advertised above, we study the cut admissibility property through semantic means, rather than the termination of a cut elimination procedure.

As is well known, standard inductive approaches do not work in higher order logic. Both for reductive cut elimination and for cut admissibility, definitions by induction over the type are required.

For this reason we adopt the Takahashi-Prawitz-Andrews approach [15, 1, 12]. These works solve the problem of possibly intensional higher-order logic with so-called V-complexes. But they do not tackle the issue of higher-order intuitionistic logic, which has been solved, independently, by Maehara [10] for the extensional version of this logic, and by Lipton-DeMarco [8] for the intensional version and constructively by Lipton-Hermant [6].

Building on these results, we define a V-complex-based model for $hoI(\mathscr{C})$, which features higher order logic, intuitionism, and constraints. Unlike the earlier works mentioned above, constraints are treated as first-class citizens and they take the place of terms, including terms of higher type. We then show that this model entails cut elimination through a completeness theorem.

A Model Theory for $hoI(\mathscr{C})$

A Kripke-style model theory for which $hoI(\mathscr{C})$ is sound and complete was introduced in [9]. The work initiated here benefits considerably from having a more algebraic (Heyting-algebra style) semantics. A complete Heyting algebra (cHa) semantics extended to treat constraints as a black box along the lines of the aforementioned Kripke semantics can easily be shown sound. The extension takes on the following form:

$\mathscr{C}$ **-soundness** If $\Gamma \vdash_{\mathscr{C}} C$ then $[\![\Gamma]\!]_\eta \leq [\![C]\!]_\eta$

$\mathscr{C}$ **-Eq** $[\![F_1 \approx F_2]\!]_\eta \wedge [\![F_1]\!]_\eta \leq [\![F_2]\!]_\eta$

$\mathscr{C}$ **-**$\exists$ $[\![\exists x^\alpha C]\!]_\eta = \bigvee\{[\![C]\!]_{\eta[x:=d]} : d \in D_\alpha\}$

where $[\![\_]\!]_\eta$ is a mapping of formulas and constraints into a cHa, η an environment mapping variables x of type α to domains D_α, C a constraint (a formula of type γ). For the full definition of such so-called *Heyting applicative structures* for ICTT see [8] and [6]. The important point about these conditions is what they say about how to treat the addition of constraints to higher-order type theory: treat truth of constraint sequents as a "black box" (just insist on soundness, do not say how that is achieved) with the exception of the treatment of existential quantification of constraints, which requires witnesses to come from the same domain that is used in the semantics of conventional formulas. Also, to enforce soundness of the (Eq) rule, we add the $\mathscr{C}$-$\exists$ rule, above.

A cut-free model theory

Proving completeness of these algebraic semantics via the so-called Lindenbaum algebra [16] requires transitivity of entailment, and so cannot be used with cut-free fragments, or in particular to show cut elimination by semantic means. A translation of Kripke semantics into Heyting algebras [16, 8] also produces a completeness result that is too weak to admit cut.

Following a blueprint laid down by Schütte in 1960 [14] and developed by Takahashi and Prawitz in 1967, one needs to develop a model theory without using cut. This technique was developed in the context of algebraic models by Maehara, and independently (in a more general context) by the authors for non-extensional intuitionistic type theory in [7]. The bulk of the authors' work at this time is to extend the Maehara technique to $hoI(\mathscr{C})$ in order to determine what restrictions on the constraint system will allow the proof to go through, in light of the fact, shown above, that cut-elimination will fail in the most general case. Interestingly, in order to show the Maehara model is a cHa, we need cut-free ivertibility of the arrow-right rule, and the existence of a constraint *of the precise same form* as in the syntactic counterexample above, namely, $B \approx (B \supset A)$ must be disallowed.

The present aim is to identify further restrictions on the constraint system that will allow the Maehara construction to go through. The counterexample above suggests that restrictions along the lines of the *uniformity* restriction in [9] which in essence only allows equations between logical formulas that have similar propositional structure may be required. This restriction is necessary in [9] to allow a semantic proof of uniformity of λProlog with constraints. It also entails the invertibility of implication-right necessary for the Maehara construction to work.

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